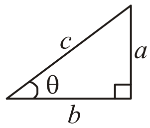
Unit 20 Applications of trigonometry

⚠ Important facts ⚠

1. Key terms

- gradient (斜率), angle of inclination (傾斜角)
- angle of elevation (仰角), angle of depression (俯角), line of sight (視線), horizontal (水平線)
- bearing (方位角), compass bearing / reduced bearing (羅盤方位角), true bearing (真方位角), whole-circle bearing (全方位角), due North (正北面)

2. Basic knowledge



(a) Trigonometric ratios:

$$\sin\theta = \frac{\text{opp. side}}{\text{hyp.}} = \frac{a}{c}, \quad a = c\cdot\sin\theta, \quad c = \frac{a}{\sin\theta}$$

$$\cos\theta = \frac{\text{adj. side}}{\text{hyp.}} = \frac{b}{c}, \quad b = c\cdot\cos\theta, \quad c = \frac{b}{\cos\theta}$$

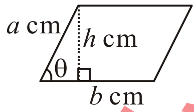
$$\tan\theta = \frac{\text{opp. side}}{\text{adj. side}} = \frac{a}{b}, \quad a = b\cdot\tan\theta, \quad b = \frac{a}{\tan\theta}$$

(b) Special angles

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

3. Areas of plane figures

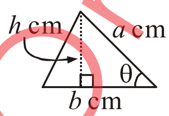
(a) If the two adjacent sides of a parallelogram are a cm and b cm long, and their included acute angle is θ, then:



the height $h = a \cdot \sin\theta$

$\therefore$ area of the parallelogram $= a \cdot b \cdot \sin\theta$

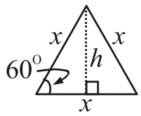
(b) If two sides of a triangle are a cm and b cm long, and their included acute angle is θ, then:



the height $h = a \cdot \sin\theta$

$\therefore$ area of the triangle $= \frac{1}{2} \cdot b \cdot a \cdot \sin\theta$

(c) If each side of an equilateral triangle is x cm, then:

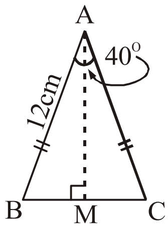


the height $h = x \cdot \sin 60°$

$\therefore$ area of the equilateral $\Delta = \frac{1}{2} \cdot x \cdot x \cdot \sin 60°$

$= \frac{1}{2}x^2 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}x^2$

(d) If the equal side and the vertical angle (頂角) or a base angle of an isosceles triangle are given, find the base and the height first.



Example: $\angle BAM = \frac{40°}{2} = 20°$

AM = 12 · cos 20°

BM = 12 · sin 20°

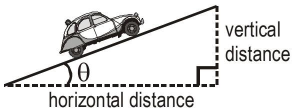
$\therefore$ area of ΔABC $= 2 \cdot (\frac{1}{2} \cdot BM \cdot AM)$

$= 12 \cos 20° \cdot 12 \sin 20°$

$= 46.3\ \text{cm}^2$ (3 sig. fig.)

4. Gradients

(a) Angle of inclination is the angle between an inclined (傾斜) plane



(or line) and the horizontal (水平線). In the figure, θ is the angle of inclination. θ is smaller than 90°.

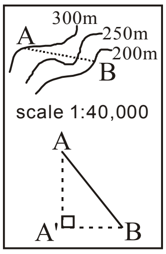
(b) Gradient = tan θ, where θ is the angle of inclination,

$\because$ gradient $= \dfrac{\text{vertical distance}}{\text{horizontal distance}}$.

(c) In problems involving contour maps (等高線地圖), if the map distance of a road is given, it refers to the *horizontal distance*, not the distance of the inclined road.

Example:

In the figure, the road AB is 5 cm on the map.



1 cm : 40000 cm = 1 cm : 400 m

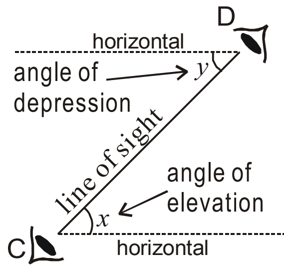
$\therefore$ A'B = 5 × 400 m = 2000 m.

AA' = 300 − 200 = 100 m.

$\therefore$ the gradient of AB $= \dfrac{100}{2000} = \dfrac{1}{20}$.

5. Angles of elevation and depression

- When a person looks at an object above him, the angle between his line of sight and the horizontal is called *the angle of elevation*. It is smaller than 90°.

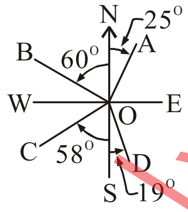


- When a person looks at an object below him, the angle between his line of sight and the horizontal is called *the angle of depression*. It is smaller than 90°.
- The angle of elevation of D from C is x, and the angle of depression of C from D is y. $x = y$ because they are alternate angles (錯角) of parallel lines.

6. Compass bearings (reduced bearings)

(a) *Compass bearings* are measured either from the north (N) or from the south (S). In the given figure:

A is N25°E from O;
B is N60°W from O;
C is S58°W from O;
D is S19°E from O.



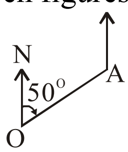
(b) A point may be due north, east, south or west of another point.

(c) NE, NW, SE and SW means that the angle is 45°.

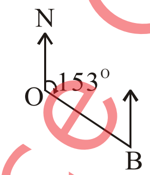
7. True bearings (whole-circle bearings)

(a) It is measured from the north in the clockwise direction.

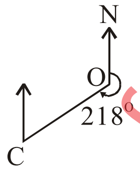
(b) In the given figures:



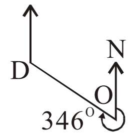
A is 050° from O,
that is, N50°E ;
O is 310° from A,
that is, S50°W.



B is 153° from O,
that is, S27°E ;
O is 333° from B,
that is, N27°W.

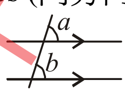


C is 218° from O,
that is, S38°W ;
O is 027° from C,
that is, N38°E.

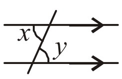


D is 346° from O,
that is, N14°W;
O is 166° from D,
that is, S14°E.

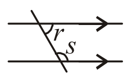
(c) In bearing problems, pay attention to the angles of parallel lines. You may need to make use of the corresponding angles (同位角), alternate angles (錯角) and interior angles (同旁內角).



$a = b$
(corr. ∠s, // lines)



$x = y$
(alt. ∠s, // lines)

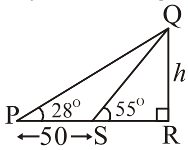


$r + s = 180°$
(int. ∠s, // lines)

8. Some special figures

In the following figures, the given length is not a side of any of the right-angled triangles:

(a)

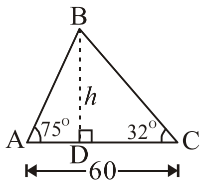


$$PR = \frac{h}{\tan 28°}, \quad SR = \frac{h}{\tan 55°}$$

$$\therefore \quad 50 = \frac{h}{\tan 28°} - \frac{h}{\tan 55°}$$

$$h = 50 \div \left(\frac{1}{\tan 28°} - \frac{1}{\tan 55°}\right) = 42.4 \text{ (3 sig. fig.)}$$

(b)



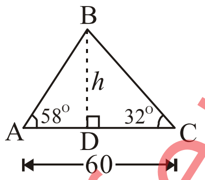
$$AD = \frac{h}{\tan 75°}, \quad DC = \frac{h}{\tan 32°}$$

$$\therefore \quad 60 = \frac{h}{\tan 75°} + \frac{h}{\tan 32°}$$

$$h = 60 \div \left(\frac{1}{\tan 75°} + \frac{1}{\tan 32°}\right)$$

$$= 32.1 \text{ (3 sig. fig.)}$$

(c)



However, in this figure:

$\angle ABC = 180° - 58° - 32° = 90°$

In ΔABC, $AB = 60 \cos 58°$

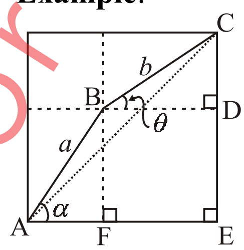
In ΔABD, $h = AB \sin 58°$

$= (60 \cos 58°)(\sin 58°)$

$= 26.96$ (4 sig. fig.)

9. In bearing problems, it is useful to make use of horizontal distances and vertical distances.

Example:



$AE = AF + FE = AF + BD$

$= a \cos \alpha + b \cos \theta$

$CE = ED + DC = FB + DC$

$= a \sin \alpha + b \sin \theta$

$$AC = \sqrt{AE^2 + CE^2}$$

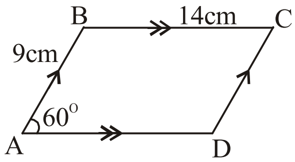
$$\tan \angle CAE = \frac{CE}{AE}$$

In this exercise, give the answers to 3 significant figures or 1 decimal place when appropriate.

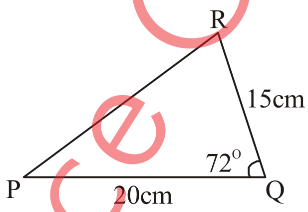
(I) Warm-up items, No.1-22 *(Time: ~60 min)*

1. Find the areas of the following figures.

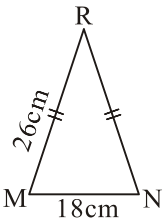
(a)



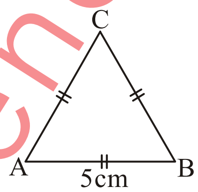
(b)



(c)



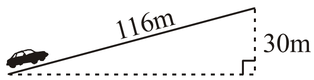
(d)



2. Find the area of a regular hexagon with side 8 cm. Express the answer in surd form.

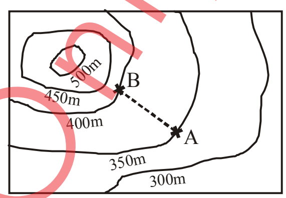
3. A road has a gradient of 1 : 15. What is its angle of inclination?

4. A car rises 30 m after travelling 116 m up a road. Find the gradient of the road.



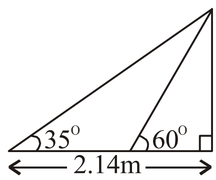
5. A car travels 15 km up a road of gradient 1 in 10 and then 20 km up another road of gradient 1 in 12. Find the vertical distance the car rises.

6. The figure shows part of a map with scale 1 : 25000. A straight road crosses the 350m and 400m contours at A and B respectively. If AB = 4cm, what is the average angle of inclination of the road?

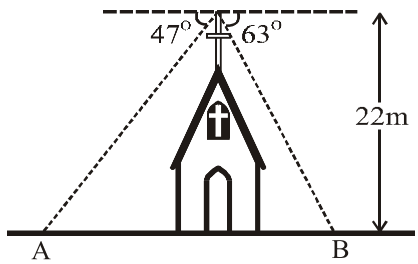


Scale 1:25000

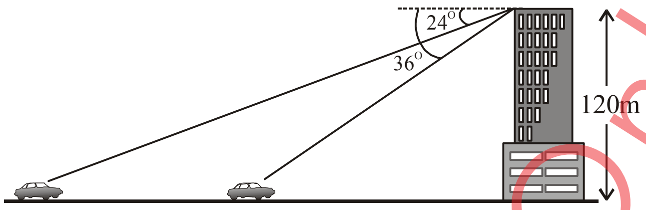
7. The length of Fiona's shadow is 2.14m when the angle of elevation of the sun is 35°. Find the length of her shadow when the angle of elevation of the sun is 60°.



8. The angle of depression from the top of a church to a point A due west of it is 47° and to a point B due east of it is 63°. If the height of the church is 22m, find the distance between A and B.



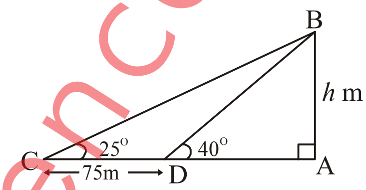
9. From the top of a building, the angles of depression of two cars on the same side of the building are 24° and 36° respectively. If the height of building is 120 m, find the distance between the cars.



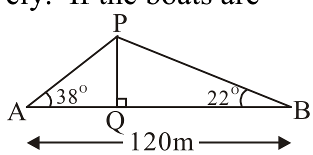
10. The angles of elevation from two ships to the top of a cliff are 25° and 40°. The ships are 75m apart.

(a) Let h m be the height of the cliff. Find AD and AC in terms of h.

(b) Find the height of the cliff to the nearest m.



11. Two boats are on opposite sides of a lighthouse. The angles of elevation of the top of the light house from the two boats are 38° and22° respectively. If the boats are 120 m apart, find the height of the lighthouse correct to the nearest integer.

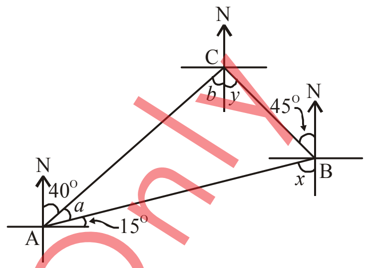


12. (a) Find the values of a, b, x and y in the given figure.

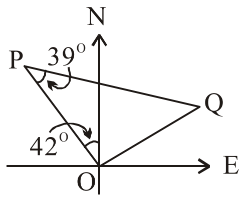
(b) Find the compass bearing of B from A.

(c) Find the true bearing of B from C.

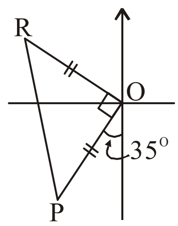
(d) Find the true bearing of C from A.



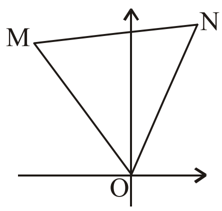
13. In the figure, P and Q are the positions of two cars. Find the bearing of Q from P.



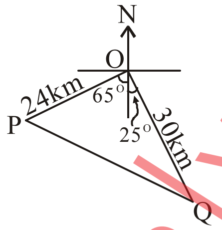
14. P is S35°W of O. If OR = OP, and ∠ROP = 90°, find the compass bearing of P from R.



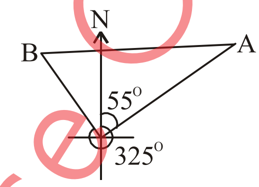
15. In the figure, OMN is an equilateral triangle. If M is 318° from O, find the true bearing of M from N.



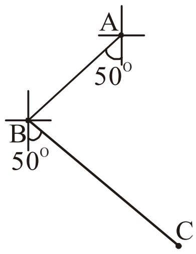
16. P is 24 km and S 65° W from O while Q is 30 km and S 25° E from O. Find the compass bearing of P from Q.



17. Car A starts travelling from a point O at 80 km/h in the direction of 055°. At the same time, car B starts travelling from O at 60 km/h in the direction of 325°. What is the distance between A and B after 2 hours?



18. Gary walked 1.5 km from A to B at on bearing of S50°W and then 3 km from B to C on a bearing of S50°E. If he walks back to A at a constant speed of 50 meter per minute, find the shortest time needed.

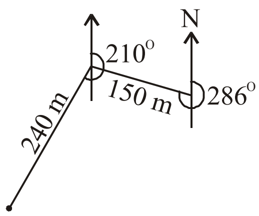


19. A man walks 150 m on a true bearing of 286° and then 240 m on a true bearing of 210°. Find his distance:

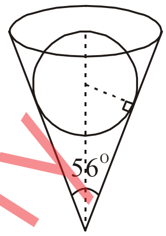
(a) west of the starting point,

(b) south of the starting point,

(c) from the starting point.



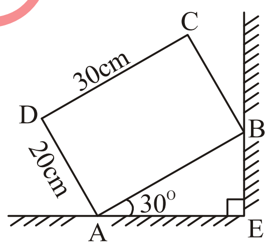
20. A sphere is put inside a sealed right cone so that the highest point of the sphere touches the base of the cone. If the height of the cone is 8 cm and its vertical angle is 56°, what is the radius of the sphere?



21. ABCD is a rectangular card with dimensions 20 cm by 30 cm. It's vertices A and B touch the edges of a table, and AB makes an angle of 30° with the horizontal line AE.

(a) Find the vertical distances from C and D to AE.

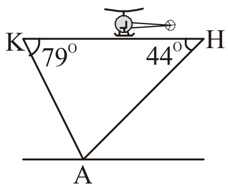
(b) Find the horizontal distance from D to BE.



22. A helicopter is flying horizontally at a certain altitude at a speed of 180 km/h. When it is at position H, its angle of depression of an airport A is 44°. After 15minutes, it comes to position K and the angle of depression of A becomes 79°.

(a) Find the distance between H and K.

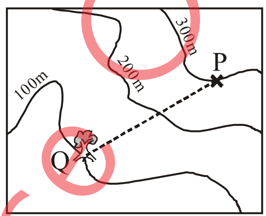
(b) Find the altitude of the helicopter.



(II) Stimulating items, No. 23-41

23. The figure shows a map with scale 1 cm : 400 m. A man standing at P is looking down at a tree Q which is 6 m tall. PQ is 4 cm long on the map, and the man's eyes are 1.6 m above his feet.

(a) There is a straight road connecting P and Q. Find the actual length of this road.

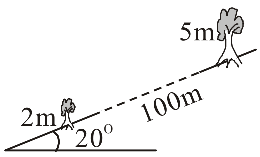


(b) Find the gradient and the angle of inclination of the road PQ.

(c) Find the angle of depression from the man to the tree.

24. From half-way up a building the angle of depression of a car is 32°. Find the angle of depression from the top of the building.

25. The figure shows two trees standing vertically on a slope of 20°. The heights of the trees are 2 m and 5 m respectively, and they are 100 m apart. Find the angle of depression from the top of the taller tree to the top of the shorter tree.



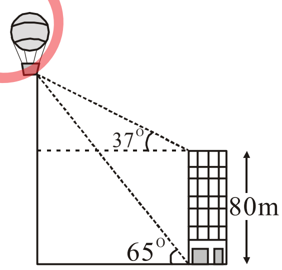
26. From a tower top, the elevation of a building is 58°, while from the tower base it is 68°. Find the height of the building if the tower is 18 m high.

27. AB and CD are two buildings of heights 65 m and h m respectively. The angle of depression of C from A is 22° and the angle of elevation of C from B is 39°.

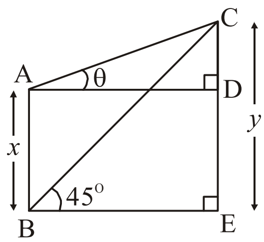
(a) Find the value of h.

(b) Find the distance between the two buildings.

28. The angles of elevation of a big balloon from the top and the bottom of a building are 37° and 65° respectively. The building is 80 m tall. Find the vertical height of the balloon.



29. AB and CE are two buildings of heights x and y respectively. The angle of elevation of C from A and B are respectively θ and 45°. Find y in terms of x and θ.

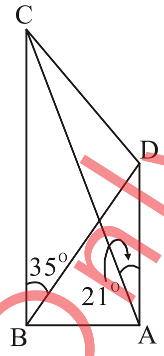


30. At 11:00 a.m., a typhoon is at 680 km S70°W of Hong Kong. It is moving in a direction of N18°E at 160 km/h.

(a) What will be its shortest distance from Hong Kong?

(b) At what time will it be nearest Hong Kong? Give your answer to the nearest minute.

31. In the figure, A is due east of B. C bears N21°W from A and is due north of B. D is due north of A and bears N35°E from B. Find the bearing of C from D.

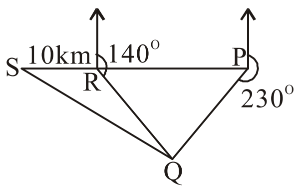


32. From a point P of a small town, a man observes that the bearing of a tower Q is 230°. He drives along a road running due west of P at a constant speed of 60 km/h. After 20 minutes, he reaches a church R from where he observes that the bearing of Q becomes 140°.

(a) Find the distance between R and Q.

(b) The man drives a further 10 km along the same road and reaches a farm S. Find the true bearing of Q from S.

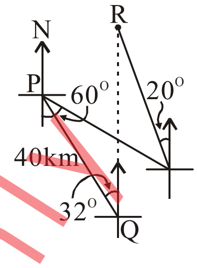
(c) If there is a straight road joining S and Q, how long will it take him to drive from S to Q at the same speed? Give the answer to the nearest 0.1 minute.



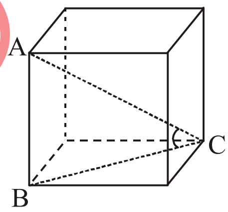
33. A lighthouse P was 40 km and N32°W from lighthouse Q. At 1:30 p.m., a ship left P and moved at a speed of 28 km/h in the direction of S60°E. When the ship was nearest to Q, it changed the course to N20°W. The ship then reached lighthouse R which is due north of Q.

(a) Find the bearing and distance of the ship from Q when the ship was nearest to it.

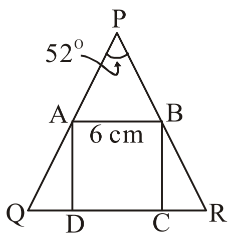
(b) At what time, correct to the nearest minute, did the ship reach R?



34. Three corners of a cube are marked A, B and C as shown. What is the value of cos ∠ACB ?



35. In the figure, ABCD is a square and ΔPQR is an isosceles triangle in which PQ = RQ. If AB = 6 cm, ∠P = 52°, find PQ.

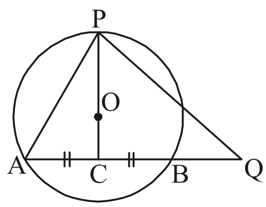


36. In the figure, O is the centre and AB is a chord of the circle, and AB is produced to Q. It is known that the radius of the circle is 5 cm, AB = 8 cm and ∠Q = 56°.

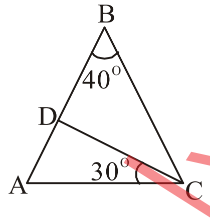
(a) If PC passes through O and C is the mid-point of AB, find ∠PCA.

(b) Find AP.

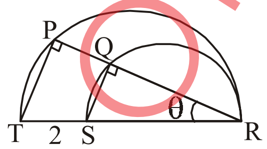
(c) Find the area of ΔAPQ.



37. In the figure, BA = BC = 12cm. If $\angle B = 40°$ and $\angle ACD = 30°$, find CD.



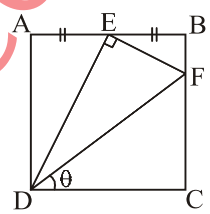
38. In the figure, TR and SR are diameters of two semicircles. and $\angle TPR = \angle SQR = 90°$. If TS = 2, find PQ in terms of θ.



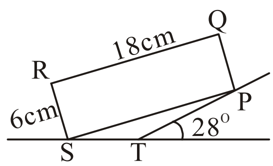
39. Given : ABCD is a square,
AE = EB and DE ⊥ EF.

(a) Prove $\Delta EAD \sim \Delta FBE$.

(b) Find the value of $\tan \theta$.



40. The figure show the cross-section of a rectangular block resting on the plane PT which inclines at an angle of 28° with the horizontal. If RS = 6 cm, RQ = 18 cm and PT = 12 cm, find the heights of Q and R above the ground.



41. (a) Find the value of y in Figure 41a.

(b) Figure 41b shows two identical lampposts, one on the horizontal level, another on a slope of 20°. The angle of elevation of the sun is 60°. Find the values of a and b.

(c) When the angle of elevation of the sun is 60°, the shadow of the lamppost on the horizontal level is 2.3m long. Find the height of the lamppost.

(d) Find the length of the shadow of the lamppost on the slope when the angle of elevation of the sun is 60°.

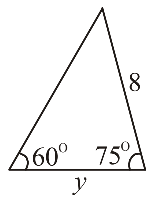


Figure 41a

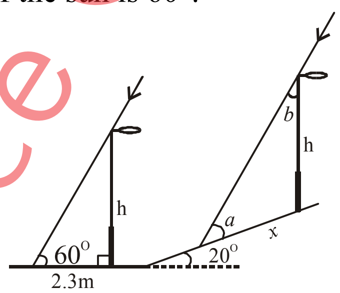


Figure 41b