ANSWERS

Unit 1 Factorization: Cross-method

1. (a)
$$\begin{array}{cc} y & +13 \\ & \times \\ y & +4 \\ \hline \multicolumn{2}{c}{+13y+4y=+17y} \end{array}$$
Ans. $(y+13)(y+4)$

(b)
$$\begin{array}{cc} a & -9 \\ & \times \\ a & -6 \\ \hline \multicolumn{2}{c}{-9a-6a=-15a} \end{array}$$
Ans. $(a-9)(a-6)$

(c)
$$\begin{array}{cc} x & +6 \\ & \times \\ x & -2 \\ \hline \multicolumn{2}{c}{+6x-2x=+4x} \end{array}$$
Ans. $(x+6)(x-2)$

(d)
$$\begin{array}{cc} y & -7 \\ & \times \\ y & +4 \\ \hline \multicolumn{2}{c}{-7y+4y=-3y} \end{array}$$
Ans. $(y-7)(y+4)$

(e) $(c+16)(c-2)$ (f) $(a+8)(a-11)$ (g) $(m-8)(m+9)$
(h) $(a+6)(a-10)$ (i) $(m-3)(m-15)$ (j) $(x+3)(x-14)$
(k) $(t+2)(t+10)$ (l) $(y+12)(y-4)$ (m) $(k-3)(k-24)$
(n) $(a-2)(a+12)$

2. (a)
$$\begin{array}{cc} 9 & -x \\ & \times \\ 7 & -x \\ \hline \multicolumn{2}{c}{-7x-9x=-16x} \end{array}$$
Ans. $(9-x)(7-x)$

(b) $=y^2+7y-44$
$$\begin{array}{cc} y & +11 \\ & \times \\ y & -4 \\ \hline \multicolumn{2}{c}{+11y-4y=+7y} \end{array}$$
Ans. $(y-4)(y+11)$

(c) $=-(m^2-18m+65)=-(m-13)(m-5)$ (d) $(18-a)(3+a)$
(e) $=-(x^2+19x+48)=-(x+3)(x+16)$ (f) $(20+y)(4-y)$
(g) $=-(n^2-21n+90)=-(n-6)(n-15)$ (h) $=50+23x-x^2=(25-x)(2+x)$

3. (a)
$$\begin{array}{cc} 3x & +2 \\ & \times \\ x & +3 \\ \hline \multicolumn{2}{c}{2x+9x=11x} \end{array}$$
Ans. $(3x+2)(x+3)$

(b)
$$\begin{array}{cc} 2y & -1 \\ & \times \\ 7y & -1 \\ \hline \multicolumn{2}{c}{-7y-2y=-9y} \end{array}$$
Ans. $(2y-1)(7y-1)$

(c)
$$\begin{array}{cc} 3x & -1 \\ & \times \\ 5x & +4 \\ \hline \multicolumn{2}{c}{-5x+12x=+7x} \end{array}$$
Ans. $(3x-1)(5x+4)$

(d)
$$\begin{array}{cc} 2y & -3 \\ & \times \\ 7y & +3 \\ \hline \multicolumn{2}{c}{-21y+6y=-15y} \end{array}$$
Ans. $(2y-3)(7y+3)$

(e) $(y+3)(12y-7)$ (f) $=40n^2+78n-27=(4n+9)(10n-3)$
(g) $(5-2x)(1-6x)$ (h) $(5m+3)(3m+5)$ (i) $(7y+2)(5y+2)$
(j) $(9-4p)(2-3p)$ (k) $(6a-5)(2a+1)$ (l) $(8m-3)(3m+4)$
(m) $(4b+5)(2b+3)$ (n) $(5a-1)(a-4)$ (o) $(9x+4)(2x-7)$
(p) $(11b-5)(3b+5)$

4. (a) $=3(x^2-11x+24)=3(x-3)(x-8)$ (b) $=2(y^2+5y-36)=2(y+9)(y-4)$
(c) $=6(2a^2+13a+15)=6(2a+3)(a+5)$ (d) $=7(3b^2+5b-12)=7(b+3)(3b-4)$
(e) $=5(4-13m+10m^2)=5(4-5m)(1-2m)$

(f) $= 6(14n^2 + 31n - 10) = 6(7n - 2)(2n + 5)$

(g) $= -4(15x^2 + 8x - 12) = -4(3x - 2)(5x + 6)$

(h) $= 3(9 + 30y + 16y^2) = 3(3 + 8y)(3 + 2y)$

5. (a)

$xy \quad -7$

$xy \quad -6$

$-7xy - 6xy = -13xy$

Ans. $(xy - 6)(xy - 7)$

(b)

$m \quad +8n$

$m \quad -3n$

$+8mn - 3mn = +5mn$

Ans. $(m - 3n)(m + 8n)$

(c) $(x + 16y)(x + 4y)$ (d) $(ab + 12)(ab - 8)$ (e) $(x^2 - 5)(x^2 + 9)$

(f) $(2 - m^2)(14 - m^2)$ (g) $(x^3 + 10)(x^3 + 5)$ (h) $(y^3 - 11)(y^3 + 6)$

(i) $(a^2 - 15b^2)(a^2 + 3b^2)$ (j) $(x^2 - 5y^2)(x^2 - 14y^2)$

(k) $= x^2 + 14x + 48 = (x + 6)(x + 8)$ (l) $= a^2 - 17a + 60 = (a - 5)(a - 12)$

6. (a) $= (5y - 3)(y - 2)$

(b) $= 23x - 12x^2 - 10 = -(12x^2 - 23x + 10) = -(4x - 5)(3x - 2)$

(c) $= 2n + 8n^2 + 3 + 12n - 18 = 8n^2 + 14n - 15 = (4n - 3)(2n + 5)$

(d) $= 24x^2 - 54x + 4x - 9 - 48x - 8 = 24x^2 - 98x - 17 = (4x - 17)(6x + 1)$

(e) $= 8y^2 - y - 15 + 15y = 8y^2 + 14y - 15 = (4y - 3)(2y + 5)$

7. (a) $(y - 2)(5y - 3)$ (b) $(5y + 2)(y - 3)$ (c) can't be factorized.

(d) can't be factorized. (e) $(a - \frac{1}{a})(a + \frac{3}{a})$

8. $= \frac{1}{9}(2x^2 - 9x - 18) = \frac{1}{9}(x - 6)(2x + 3)$

9. $= (x^2 + 3)(x^2 - 9) = (x^2 + 3)(x + 3)(x - 3)$

10. $= (x^2 - 4y^2)(4x^2 - 9y^2) = (x + 2y)(x - 2y)(2x + 3y)(2x - 3y)$

11. $= 2a(x^2 + 3x + 2) + 3b(x^2 + 3x + 2) = (2a + 3b)(x^2 + 3x + 2) = (2a + 3b)(x + 2)(x + 1)$

12. (a) $(6a - 1)(2a + 5)$

(b) Put $2x + 1 = a$ into part (a), we get:
$[6(2x + 1) - 1][2(2x + 1) + 5] = (12x + 5)(4x + 7)$

13. (a) $(x - 6y)(x + 2y)$

(b) $= 3[b^2 - 4b(a + 1) - 12(a + 1)^2]$.
Put $b = x, a + 1 = y$ into part (a), we get:
$3[b - 6(a + 1)][b + 2(a + 1)] = 3(b - 6a - 6)(b + 2a + 2)$

14. (a) $(x + 12)(x + 2)$

(b) $= (a + 2b)^2 + 14(a + 2b) + 24$. Put $a + 2b = x$ into part (a),
we get: $[(a + 2b) + 12][(a + 2b) + 2] = (a + 2b + 12)(a + 2b + 2)$

15. (a) $= \frac{1}{4}(3y^2 - 8y + 4) = \frac{1}{4}(3y - 2)(y - 2)$

(b) $= 3m - n + \frac{3}{4}(m + n)^2 - 5m - n + 1 = \frac{3}{4}(m + n)^2 - 2m - 2n + 1 = \frac{3}{4}(m + n)^2 - 2(m + n) + 1$
Put $m + n = y$ into part (a), we get:
$\frac{1}{4}[(3(m + n) - 2][(m + n) - 2] = \frac{1}{4}(3m + 3n - 2)(m + n - 2)$

16. (a) $= x(y + 8) - 2(y + 8) = (x - 2)(y + 8)$

(b) $a^2 - b^2 = (a + b)(a - b)$, let $x = a - b$, $y = a + b$,

$\therefore\ 8x - 2y = 8(a - b) - 2(a + b) = 8a - 8b - 2a - 2b = 6a - 10b$

$\therefore$ From (a), $a^2 - b^2 + 6a - 10b - 16 = [(a - b) - 2][(a + b) + 8] = (a - b - 2)(a + b + 8)$

17. (a) $(7)(-12) = -84,\ 7 + (-12) = -5,\quad \because\ m > n,\quad \therefore\ m = 7,\ n = -12.$

(b) $x^3 + 8x^2 - 5x - 84 = x^3 + 7x^2 + (x^2 - 5x - 84) = x^2(x + 7) + (x + 7)(x - 12)$

$= (x + 7)[x^2 + (x - 12)] = (x + 7)(x + 4)(x - 3)$

18. (a) $8a^2 + 18ab - 5b^2 = (4a - b)(2a + 5b)$

(b) $8a - 2b - 8a^2 - 18ab + 5b^2$

$= (8a - 2b) - (8a^2 + 18ab - 5b^2)$

$= 2(4a - b) - (4a - b)(2a + 5b)$

$= (4a - b)[2 - (2a + 5b)] = (4a - b)(2 - 2a - 5b)$

19. (a) $36h^2 - 25 = (6h - 5)(6h + 5)$

(b) $6h^2k + 37hk - 35k = k(6h^2 + 37h - 35) = k(6h - 5)(h + 7)$

(c) $6h^2k - 36h^2 + 37hk - 35k + 25k^2$

$= 6h^2k + 37hk - 35k - 36h^2 + 25k^2$

$= (6h^2k + 37hk - 35k) - (36h^2 - 25k^2)$

$= k(6h - 5)(h + 7) - (6h - 5)(6h + 5)$

$= (6h - 5)[k(h + 7) - (6h + 5)] = (6h - 5)(hk + 7k - 6h - 5)$

20. $16a^4 + (9 - 6a + 5a^2)(9 - 6a - 3a^2)$

$= 16a^4 + (9 - 6a + a^2 + 4a^2)(9 - 6a + \underline{\ a^2\ } - \underline{4a^2})$

$= 16a^4 + [(3 - a)^2 + 4a^2][(\ \underline{\ 3 - a\ }\ )^2 - \underline{\ 4a^2\ }\ ]$

$= \underline{\ 16a^4 + (3 - a)^4 - 16a^4\ }$

$= \underline{\ (3 - a)^4\ }$

21. (a) $63 + (u - 5)(u - 21) = 63 + u^2 - 26u + 105 = u^2 - 26u + 168 = (u - 14)(u - 12)$

(b) $63 + (x + 1)(x + 3)(x - 5)(x - 7) = 63 + (x + 1)(x - 5)(x + 3)(x - 7)$

$= 63 + (x^2 - 4x - 5)(x^2 - 4x - 21) = 63 + [(x^2 - 4x) - 5][(x^2 - 4x) - 21]$

$= [(x^2 - 4x) - 14][(x^2 - 4x) - 12]$ [from (a)]

$= (x^2 - 4x - 14)(x^2 - 4x - 12) = (x^2 - 4x - 14)(x - 6)(x + 2)$

Unit 2 Laws of indices

1. (a) $= 4x^{8-2} = 4x^6$ (b) $= \dfrac{9 \times 3}{6} y^{3+6-1} = \dfrac{9}{2} y^8$ (c) $= -3 \times p^{2+2+2} \times q^{1+2} = -3p^6q^3$

(d) $= \dfrac{4k^2 \times 2k^3}{-16k^5} = -\dfrac{1}{2} k^{2+3-5} = -\dfrac{1}{2}$ (e) $= \dfrac{(-5)^2 a^4 b^2 c^2}{(-bc^2)(-5)^3 a^9} = \dfrac{b^{2-1} c^{2-2}}{5a^{9-4}} = \dfrac{b}{5a^5}$

(f) $= \dfrac{(-2)(-15)}{-\frac{1}{5}} r^{2-1+3} s^{3-6+12} = -150 r^4 s^9$

(g) $= \dfrac{a^6b^8 \times 36a^4b^6}{-12a^2b} = -3a^{6+4-2}b^{8+6-1} = -3a^8b^{13}$

(h) $= \dfrac{x^8y^4 \times 9x^2y^4}{-81x^{10}y^2} = -\dfrac{x^{8+2-10}y^{4+4-2}}{9} = -\dfrac{y^6}{9}$

(i) $= (\dfrac{3^2b^2}{2a^2})^{-2}(\dfrac{3a^4}{2^2b^6})^3 = \dfrac{3^{-4}b^{-4}}{2^{-2}a^{-4}} \cdot \dfrac{3^3a^{12}}{2^6b^{18}} = \dfrac{3^{-1}a^{16}}{2^4b^{22}} = \dfrac{a^{16}}{48b^{22}}$

(j) $= \dfrac{-6m^2np^2}{(n^4p^2)(mn^2p)} \times \dfrac{(-3mn)(25n^4)}{10m^3np} = 45m^{2-1+1-3}n^{1-4-2+1+4-1}p^{2-2-1-1} = \dfrac{45}{mnp^2}$

2. (a) $= -(1) \times 8 = -8$ (b) $= (1) \times 81 = 81$ (c) $-\dfrac{1}{7}$

(d) $= -\dfrac{1}{6^2} = -\dfrac{1}{36}$ (e) $= (\dfrac{3}{5})^{-3} = \dfrac{125}{27}$ (f) $= 2 - \dfrac{1}{8} = \dfrac{15}{8}$

(g) $= -1 \times 13^{-5} \times 13^4 = -\dfrac{1}{13}$ (h) $= -16 - \dfrac{1}{16} + 4 = \dfrac{-12 \times 16 - 1}{16} = \dfrac{-193}{16}$

(i) $= (\dfrac{4}{3})^{-3-(-2)-(-2)} = \dfrac{4}{3}$ (j) $= \dfrac{2^{2(11)}}{2^{23}} + \dfrac{1}{4} = \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{3}{4}$

(k) $= [-(\dfrac{1}{2})^2 + \dfrac{1}{3}] \div (\dfrac{1}{5})^2 = (\dfrac{-3}{12} + \dfrac{4}{12}) \times \dfrac{25}{1} = \dfrac{25}{12}$

3. (a) $= \dfrac{2}{x} \div \dfrac{3x}{y^3} = \dfrac{2}{x} \times \dfrac{y^3}{3x} = \dfrac{2y^3}{3x^2}$ (b) $= \dfrac{a^2}{6} \times \dfrac{2}{b^3} = \dfrac{a^2}{3b^3}$

(c) $= a^4 \times \dfrac{-3}{a^3} \times a = -3a^2$ (d) $= \dfrac{-36a^{2+5}b^{-2-6}}{6(1)} = \dfrac{-6a^7}{b^8}$

(e) $= \dfrac{x^{-9}y^{-6}}{2^3} \div \dfrac{1}{(-4x^4)^2} = \dfrac{1}{2^3x^9y^6} \times 2^4x^8 = \dfrac{2}{xy^6}$

(f) $= 4p^4q^{-6} \times p^{-3}q^{-2} \times 3p^{-1}q^2 = 12p^{4-3-1}q^{-6-2+2} = \dfrac{12}{q^6}$

(g) $= \dfrac{36a^{-3}b^2}{-8a^2b} \times (-24^3a^6b^{-12}) = \dfrac{24^3 \times 36}{8}a^{-3-2+6}b^{2-1-12} = \dfrac{62208a}{b^{11}}$

(h) $= (\dfrac{8x^{-1}y^{-1}}{3})^{-2} \cdot \dfrac{2^3x^{-12}y^{12}}{(-15)x^3y^{-3}} = \dfrac{8^{-2}x^2y^2}{3^{-2}} \cdot \dfrac{2^3y^{15}}{-15x^{15}} = -\dfrac{9 \cdot 8y^{17}}{8^2 \cdot 15x^{13}} = -\dfrac{3y^{17}}{40x^{13}}$

4. (a) $10^{y-3} = 10^0$, $\therefore$ $y - 3 = 0, y = 3$ (b) $9^{4x-1} = 9^2$, $\therefore$ $4x - 1 = 2, x = \dfrac{3}{4}$

(c) $4^{5y} = 4^{-3}$, $\therefore$ $5y = -3, y = -\dfrac{3}{5}$ (d) $5^{2n+1} = 5^{-3}$, $\therefore$ $2n + 1 = -3, n = -2$

(e) $2^{-n} = 2^5$, $\therefore$ $-n = 5, n = -5$ (f) $y^{-2} = (\dfrac{6}{7})^2 = (\dfrac{7}{6})^{-2}$, $\therefore$ $y = \dfrac{7}{6}$

(g) $x^{-3} = \dfrac{27}{8} = (\dfrac{3}{2})^3 = (\dfrac{2}{3})^{-3}$, $\therefore$ $x = \dfrac{2}{3}$

(h) $3^{3(x+2)} = 3^{2(x+1)}$, $\therefore$ $3(x+2) = 2(x+1)$, $3x + 6 = 2x + 2$, $x = -4$

(i) $11^{x+3-2x} = 11^0$, $\therefore$ $x + 3 - 2x = 0$, $x = 3$

(j) $5\times 3^{2+x}=5\times 27$, $5\times 3^{2+x}=5\times 3^3$, $\therefore$ $2+x=3$, $x=1$

(k) $5^2\times 5^{2x-8}=1$, $5^{2+2x-8}=5^0$, $\therefore$ $2+2x-8=0$, $2x-6=0$, $x=3$

5. (a) $=\dfrac{2^{3x}\times 2^{2(x+1)}}{2^{5x}} = 2^{3x+2x+2-5x} = 2^2 = 4$

(b) $=\dfrac{2^{n+1}\times 3^n}{2^{n-1}\times 3^{n-1}} = 2^{n+1-(n-1)}\times 3^{n-(n-1)} = 2^2\times 3^1 = 12$

(c) $=\dfrac{5^n\times 3^n}{5^{n+1}\times 3^{n-2}} = 5^{n-(n+1)}\times 3^{n-(n-2)} = 5^{-1}\times 3^2 = \dfrac{9}{5}$

(d) $= 7^{-2x}\times\dfrac{7^{x+1+x-1}}{7^{x(x-1)+x}} = 7^{-2x+2x-[x(x-1)+x]} = \dfrac{1}{7^{x^2}}$

6. (a) 3.48×10^{-5} (b) 2.50×10^{11} (c) -9.42×10^8

(d) $=[7.10\times 10^{-1}\div(6.29\times 10^7)]^3 = (\dfrac{7.10}{6.29})^3\times 10^{(-1-7)\times 3} = 1.44\times 10^{-24}$

7. (a) $=\sqrt{49\times 10^{-16}} = \sqrt{(7\times 10^{-8})^2} = 7.00\times 10^{-8}$

(b) $=(28\times 10^{-11})\div(5\times 10^3) = 5.60\times 10^{-14}$

(c) $=2.3\times 10^8+0.19\times 10^8 = (2.3+0.19)\times 10^8 = 2.49\times 10^8$

(d) $=0.4\times 10^{-7}-2\times 10^{-7} = (0.4-2)\times 10^{-7} = -1.60\times 10^{-7}$

8. (a) $2^k=\dfrac{1}{8}=\dfrac{1}{2^3}=2^{-3}$, $\therefore$ $k=-3$

(b) $x^{-3}=\dfrac{91}{125}+1=\dfrac{216}{125}=(\dfrac{6}{5})^3=(\dfrac{5}{6})^{-3}$, $\therefore$ $x=\dfrac{5}{6}$

(c) $(2^{-2})^{3x-1}=2^{3(2x+4)}$, $2^{-6x+2}=2^{6x+12}$, $\therefore$ $-6x+2=6x+12$, $-12x=10$, $x=-\dfrac{5}{6}$

(d) $5^{n+1}\times 5^{4n}\times 5^{-9n}=5^{-4}$, $5^{1-4n}=5^{-4}$, $\therefore$ $1-4n=-4$, $n=\dfrac{5}{4}$

(e) $7^{2x}\times 7^{x+3}=7^{-1}$, $7^{3x+3}=7^{-1}$, $\therefore$ $3x+3=-1$, $x=-\dfrac{4}{3}$

(f) $3\times 5^{6-x}=75$, $3\times 5^{6-x}=3\times 5^2$, $\therefore$ $6-x=2$, $x=4$

9. (a) $=6^{n+1}-6^n = 6^n(6-1) = 5\times 6^n$

(b) $=7^{n-2}(7^3-6) = 337\times 7^{n-2}$

(c) $=2^{3(n+1)}-2^{3n-1} = 2^{3n+3}-2^{3n-1} = 2^{3n-1}(2^4-1) = 15\times 2^{3n-1}$

(d) $=\dfrac{2^{n-2}(1-2^4)}{2^{n-2}\times 2^3} = -\dfrac{15}{8}$ (e) $=\dfrac{2\times 3^{n-1}-4\times 3^{n-1}}{3^{n-1}\cdot 3^2-3^{n-1}\cdot 3} = \dfrac{3^{n-1}(2-4)}{3^{n-1}(3^2-3^1)} = -\dfrac{2}{6} = -\dfrac{1}{3}$

(f) $=\dfrac{3\times 2\times 2^{2(3n-2)}+2\times 2^{3(2n)}}{2^{2(3n)}+2^{3(2n-1)}} = \dfrac{3\times 2^{6n-3}+2^{6n+1}}{2^{6n}+2^{6n-3}} = \dfrac{2^{6n-3}(3+2^4)}{2^{6n-3}(2^3+1)} = \dfrac{19}{9}$

10. (a) $3^x(3-1)=2\times 81$, $3^x=3^4$, $\therefore$ $x=4$

(b) $10\times 2^{n-1}+2^{n-1}=88$, $2^{n-1}(10+1)=88$, $2^{n-1}=2^3$, $\therefore$ $n-1=3$, $n=4$

(c) $4^x+4^{x-1}=80$, $4^{x-1}(4+1)=80$, $4^{x-1}=4^2$, $\therefore$ $x-1=2$, $x=3$

(d) $3^{n-1}(3^2-3+1)=7$, $3^{n-1}=1$, $3^{n-1}=3^0$, $\therefore$ $n-1=0$, $n=1$

(e) $5^{2y+1}-5^{2y}=\frac{4}{5}$, $5^{2y}(5-1)=4\times5^{-1}$, $4\times5^{2y}=4\times5^{-1}$, $\therefore\ 2y=-1$, $y=-\frac{1}{2}$

(f) $2^{2x}-2^{2x+3}+\frac{7}{4}=0$, $2^{2x}(2^3-1)=\frac{7}{4}$, $2^{2x}=2^{-2}$, $\therefore\ 2x=-2, x=-1$

11. (a) $=\frac{1}{2a^{-1}+b^{-1}}\times\frac{ab}{ab}=\frac{ab}{2b+a}$ (b) $=\frac{1}{2x^{-1}-3y^{-1}}\times\frac{xy}{xy}=\frac{xy}{2y-3x}$

(c) $=(\frac{1}{r}+\frac{5}{s})^{-2}=(\frac{s+5r}{rs})^{-2}=\frac{r^2s^2}{(s+5r)^2}$

(d) $=\frac{1}{(m-n)^2}\times(\frac{1}{m^{-2}-n^{-2}}\times\frac{m^2n^2}{m^2n^2})=\frac{1}{(m-n)^2}\times\frac{m^2n^2}{n^2-m^2}$

$=\frac{m^2n^2}{(n-m)^2(n-m)(n+m)}=\frac{m^2n^2}{(n-m)^3(n+m)}$

12. $(2^{x+3})^2=10^2$, $2^{x+3}=10$, $2^x\cdot2^3=10$, $2^x=\frac{10}{8}=\frac{5}{4}$

13. $(3^{n+2})^2=36=6^2$, $3^{n+2}=6$, $3^{n+1}\times3=6$, $\therefore\ 3^{n+1}=2$

14. $2^{2(y-2x)}=2^{-2}$, $(2^{y-2x})^2=(2^{-1})^2$, $\therefore\ y-2x=-1$, $\therefore\ y=2x-1$ …(i).

Sub (i) into $9^{x+y}-3^{4y-x}=0$, we have

$9^{x+(2x-1)}-3^{4(2x-1)-x}=0$, $3^{2(3x-1)}=3^{7x-4}$, $\therefore\ 6x-2=7x-4$, $x=2$.

Sub $x=2$ into (i), we have $y=2(2)-1=3$. $\therefore\ x=2$ and $y=3$

15. $=[\frac{\sqrt{x}(\sqrt{x}-\sqrt{y})+(\sqrt{y}(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}]^{-1}=(\frac{x-\sqrt{xy}+\sqrt{xy}+y}{x-y})^{-1}=(\frac{x+y}{x-y})^{-1}=\frac{x-y}{x+y}$

16. (a) $(x-x^{-1})^2=(\frac{9}{20})^2$, $x^2-2x(x^{-1})+x^{-2}=\frac{81}{400}$, $\therefore\ x^2+x^{-2}=\frac{81}{400}+2=2\frac{81}{400}$

(b) $(x+x^{-1})^2=x^2+2(x)(x^{-1})+x^{-2}=x^2+x^{-2}+2=2\frac{81}{400}+2=4\frac{81}{400}$,

$\therefore\ x+x^{-1}=\sqrt{4\frac{81}{400}}=\sqrt{\frac{1681}{400}}=\frac{41}{20}$, $\frac{x}{2}+\frac{x^{-1}}{2}=\frac{1}{2}\times\frac{41}{20}=\frac{41}{40}$

17. (a) 1cm = 10^{-2} m = $10^{-2}\times10^9$ nanometer = 10^7 nanometer

$\therefore$ 126 cm = 126 × 10^7 nanometer = 1.26 × 10^9 nanometer

(b) 0.80km = 0.80 × 10^3 m = 0.80 × 10^3 × 10^9 nanometer

= 8.0 × 10^{11} nanometer

18. (a) The lower limit = 3.95 × 10^5 m = 3.95 × 10^2 km = 395 km.

The upper limit = 4.05 × 10^5 m = 4.05 × 10^2 km = 405 km.

(b) The relative error = $\frac{0.05\times10^5}{4.00\times10^5}=\frac{1}{80}$; the percentage error = $\frac{1}{80}$ × 100% = 1.25%

(c) 4.00 × 10^5 m means that the length is accurate to 3 significant figures. But we don't know the number of significant figures in 400 000 m, and without knowing the degree of accuracy, we can't find the maximum percentage error.

19. Let the thickness of the oil layer be *t* mm. $(40\times10)^2\pi t=4$, $t=\frac{4}{(16\times10^4)\pi}=7.96\times10^{-6}$ mm.

Ans. Thickness of the oil layer is 7.96×10 $^{-6}$ *mm.*

20. (a) Speed = $300\,000\,000 \div 1\,000 \times 3\,600 = 1\,080\,000\,000 = 1.08 \times 10^9$ km/hr
    (b) Distance = $(1.08 \times 10^9) \times 24 \times 365 \times 100 = 9.46 \times 10^{14}$ km

21. The no. of years for the light from star A47 to reach the Earth = $\dfrac{1.32 \times 10^{15}}{9.46 \times 10^{12}}$
    = 140 years (3 sig. fig.).
    2018 – 140 = 1878 (the year); at that time the star A47 still existed,
    $\therefore$ it is possible for us to see it in the year 2018.

22. (a) $N = 8^{21} + 4^{17} - 2^{33} = (2^3)^{21} + (2^2)^{17} - 2^{33} = 2^{63} + 2^{34} - 2^{33}$
    $= 2^{63} + 2 \times 2^{33} - 2^{33} = 2^{63} + 2^{33}$
    (b) $N = 2^{15\times4+3} + 2^{8\times4+1} = 2^3 \times (2^4)^{15} + 2^1 \times (2^4)^8 = 8 \times 16^{15} + 2 \times 16^8$

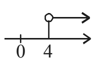
23. (a) (i) 20 038
    (ii) $20\,038 = 2 \times 10^4 + 0 \times 10^3 + 0 \times 10^2 + 3 \times 10^1 + 8 \times 10^0$
    (b) $10\,0100\,1000\,0001_2 = 1 \times 2^{13} + 1 \times 2^{10} + 1 \times 2^7 + 1 \times 2^0 = 9\,345 < 20\,038$

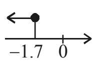
24. (a) $a \times 10^n - b \times 10^{n+3} = a \times 10^n - b \times 10^3 \times 10^n = (a - b \times 10^3) \times 10^n$
    $\therefore\ c = a - b \times 10^3 = a - 1000b$
    (b) $2.3 \times 10^{-321} - 0.8 \times 10^{-158} \times 4.3 \times 10^{-160}$
    $= 2.3 \times 10^{-321} - 0.8 \times 4.3 \times 10^{-158} \times 10^{-160}$
    $= 2.3 \times 10^{-321} - 3.44 \times 10^{-318} = 2.3 \times 10^{-321} - 3.44 \times 10^{-321+3}$
    $= (2.3 - 1\,000 \times 3.44) \times 10^{-321} = -3\,437.7 \times 10^{-321} = -3.4377 \times 10^3 \times 10^{-321}$
    $= -3.4377 \times 10^{-318}$

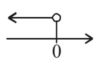
25. (a) $777^{20} = 6.4331 \times 10^{57}$ (5 sig. fig.)
    (b) $777\,000^{800} = [(777 \times 10^3)^{20}]^{40} = (777^{20} \times 10^{60})^{40} = (6.4331 \times 10^{57} \times 10^{60})^{40}$ [from (a)]
    $= 6.4331^{40} \times 10^{4680} = 2.17 \times 10^{32} \times 10^{4680} = 2.17 \times 10^{4712}$ (3 sig. fig.)

26. (a) $92\,263\,734\,836 = 9.226\,373\,483\,6 \times 10^{10} = 9.23 \times 10^{10}$ (3 sig. fig.)
    (b) Number of different combinations that can be generated in 1 day
    $= (1.6 \times 10^6) \times (24 \times 60 \times 60) = 1.3824 \times 10^{11} > 9.23 \times 10^{10}$
    $\therefore$ The claim is agreed.

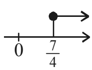
Unit 3 Inequalities

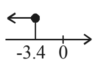
1. (a) 

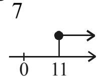
   (b) 

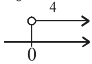
   (c) 

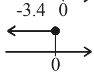
2. (a) $x > -2$ (b) $x \le 1.5$ (c) $x \ge 0$ (d) $x < -3\dfrac{1}{7}$

3. (a) $x \ge \dfrac{7}{4}$ 

   (b) $x \le -3.4$ 

   (c) $x \ge 11$ 

   (d) $x > 0$ 

   (e) $x \le 0$ 

4. (a) $x \le 13$ (b) $y \le 7$ (c) $5x + 3 > 13$
   (d) $\dfrac{y}{2} - 2 \le 7$ (e) $\dfrac{x+10}{4} \ge 8$ (f) $\dfrac{2}{5}m - 1 \le 11$

5. (a) $-9x < 9, x > -1$ (b) $4x \ge -14, \quad \therefore \quad x \ge -\frac{7}{2}$

(c) $-8x \le 0, \quad \therefore \quad x \ge 0$ (d) $7 - x < 7, \quad -x < 0, \quad \therefore x > 0$

(e) $6x - 9 \le 4x, \quad 2x \le 9, \quad \therefore \quad x \le \frac{9}{2}$

(f) $-20x + 5 > 2x - 6, \quad -22x > -11, \quad \therefore \quad x < \frac{1}{2}$

(g) $12x + 8 \ge 3x + 14 - 7x, \quad 16x \ge 6, \quad \therefore \quad x \ge \frac{3}{8}$

(h) $5 - 4x < 8 - 6x + 3, \quad 2x < 6, \quad \therefore \quad x < 3$

(i) $2(5x - 7x - 14) > 12 - 3x, \quad 2(-2x - 14) > 12 - 3x,$
$-4x - 28 > 12 - 3x, \quad -x > 40, \quad \therefore \quad x < -40$

6. (a) $3x + 4 < 54 - 12x, \quad 15x > 50, \quad \therefore \quad x < \frac{10}{3}$

(b) $3(3x + 4) \ge 5(2 - x), \quad 9x + 12 \ge 10 - 5x, \quad 14x \ge -2, \quad \therefore \quad x \ge -\frac{1}{7}$

(c) $(5x - 1) - 9 \le 0, \quad 5x \le 10, \quad \therefore \quad x \le 2$

(d) $(8x + 12) - 5 > 50x, \ -42x > -7, \quad \therefore \quad x < \frac{1}{6}$

(e) $4(2x + 11) + 3(6 - x) \ge 12, \quad 8x + 44 + 18 - 3x \ge 12, \quad 5x \ge -50, \quad \therefore \quad x \ge -10$

(f) $-3(x - 5) + 24 < 2(2x - 3), \quad -3x + 15 + 24 < 4x - 6, \quad -7x < -45, \quad \therefore \quad x > \frac{45}{7}$

(g) $x + 6 - 2(x - 3) \le 20, \quad x + 6 - 2x + 6 \le 20, \quad -x \le 8, \quad \therefore \quad x \ge -8$

(h) $6(1 + 2x) - 2(4x + 7) < 6 + 12x - 8x - 14 < 27 - 3x,$
$4x - 8 < 27 - 3x, \quad 7x < 35, \quad \therefore \quad x < 5$

7. $2(15 + x) \le 18, \quad 15 + x \le 9, \quad x \le -6.$ *Ans. The greatest value of x is –6.*

8. $\frac{y}{3} + 13 \le y, \quad y + 39 \le 3y, \quad -2y \le -39, \quad y \ge 19.5.$ *Ans. The least value of y is 20.*

9. Let x be the smallest integer. $x + (x + 1) + (x + 2) < 15, \quad 3x < 12, \quad x < 4.$
Ans. The maximum value of the smallest number is 3.

10. Let x be the larger odd number. $x + (x - 2) > 28, \quad 2x > 30, \quad x > 15$
Ans. The least value of the larger odd number is 17.

11. Let h cm be David's height. $h + (h - 14) \ge 280, \quad 2h \ge 294, \quad \therefore h \ge 147.$
Ans. The height of David is at least 147 cm.

12. Let x be the number of hotdogs. $16x + 8.4 \times 5 \le 150, \quad 16x \le 108, x \le 6.75.$
Ans. She can buy 6 hotdogs at most.

13. Let x be the number of \$2 coins.
$2x + 0.5(x - 8) < 56, \quad 2x + 0.5x - 4 < 56, \quad 2.5x < 60, \quad x < 24.$
Ans. The maximum number of \$2 coins is 23.

14. $2(y + 15) > 3y, \quad 2y + 30 > 3y, \quad 30 > y, \quad y < 30.$ Besides, y must be a positive number.
Ans. y must be a positive number smaller than 30.

15. Let x be the number of incorrect answers.
$3(20 - x) - 2x > 50, \quad 60 - 3x - 2x > 50, \quad -5x > -10, \quad x < 2.$
Ans. The maximum number of incorrect answers is 1.

16. $2(9 + a) \ge 40, \quad 9 + a \ge 20, \quad a \ge 11$
Minimum area = Minimum width × 9 = 11 × 9 = 99 cm^2

17. Let x be the smaller number. $x > \frac{x+4}{2}$, $2x > x+4$, $x > 4$; and x must be a multiple of 4.
Ans. *The least value of the smaller number is 8.*

18. Let x be the largest number.
$x+(x-3)+(x-6) \le 30$, $3x \le 39$, $x \le 13$; and x must be a multiple of 3.
Ans. *The greatest value of the largest number is 12.*

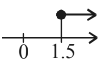
19. (a) Let x be the smaller number. $x + (x + 7) < 19$, $2x < 12$, $\therefore$ $x < 6$.
Ans. *The smaller number is smaller than 6.*
(b) $\because$ $0 < x < 6$, $\therefore$ $7 < x + 7 < 13$, and x is an integer.
Ans. *The possible values of the larger number are 8, 9, 10, 11 and 12.*

20. $a < -3$, $\therefore$ $a + b < b - 3$……(i);
$b < 15$, $b - 3 < 12$……….(ii);
From (i) and (ii), $a + b < b - 3 < 12$, $\therefore$ $a + b < 12$

21. (a) Let $a = \frac{1}{2}$, $a^2 = (\frac{1}{2})^2 = \frac{1}{4}$, $a^2 < a$. $\therefore$ The statement is not correct.
(b) If $a = -5$, $b = -3$, $a < b$, but $a^2 = (-5)^2 = 25$, $b^2 = (-3)^2 = 9$, $a^2 < b^2$
$\therefore$ The statement is not correct.
(c) If $c = 2$, $d = -2$, $c < d$, but $\frac{1}{c} = \frac{1}{2}$, $\frac{1}{d} = -\frac{1}{2}$, $\frac{1}{c} > \frac{1}{d}$
$\therefore$ The statement is not correct.
(d) If $a = -4$, $b = 3$, $c = -6$, $d = -2$, $a < b$, and $c < d$,
but $ac = (-4)(-6) = 24$, $bd = 3(-2) = -6$, $a \times c > b \times d$.
$\therefore$ The statement is not correct.

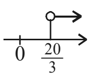
22. (a) $6x+9-1-5x > x-5$, $x + 8 > x - 5$, $8 > -5$,
Ans. *x can be any real numbers.*
(b) $-\frac{3-4x}{2} > 2x-1$, $-3+4x > 4x-2$, $-3 > -2$ *Ans. There is no solution.*

23. (a) $0.81-3.24x+6.97x \le 1.05(3x+1.6x-0.8)$,
$0.81+3.73x \le 1.05(4.6x-0.8)$



$0.81+3.73x \le 4.83x-0.84$, $-1.1x \le -1.65$, $x \ge \frac{1.65}{1.1}$, $\therefore$ $x \ge 1.5$

(b) $60\times\frac{1}{3}[-\frac{9}{4}+\frac{8x}{5}+\frac{1}{4}(9x-1)] > 60\times(\frac{6x-1}{5}+\frac{6-x}{6}+\frac{1}{30})$



$20(-\frac{9}{4}+\frac{8x}{5}+\frac{9x}{4}-\frac{1}{4}) > 12(6x-1)+10(6-x)+2$
$-45 + 32x + 45x - 5 > 72x - 12 + 60 - 10x + 2$
$77x - 50 > 62x + 50$, $15x > 100$, $\therefore$ $x > \frac{20}{3}$

24. (a) Not true. The product of two negative numbers must be positive.
(b) True. $\because$ $a < b$, $\therefore$ $a + 5 < b + 5$, but $b + 5 < b + 6$, $\therefore$ $a + 5 < b + 6$.
(c) True. $kx + h^2 < hx + k^2$, $h^2 - k^2 < hx - kx$, $(h + k)(h - k) < (h - k)x$,
$\because$ $h - k > 0$, $\therefore$ $h + k < x$, $x > h + k$

25. (a) Not true. For example, when $p = -1$, $q = -2$, $(-1) + (-2) = -3 < 0$.
(b) True. $\because$ $q < p$, $\therefore$ $q - p < p - p$, i.e. $q - p < 0$.
(c) True. $\because$ $8 > p > q > -6$, $\therefore$ $8 > p$ and $q > -6$, i.e. $0 > (p - 8)$ and $(q + 6) > 0$. Since $(p - 8)$ is negative and $(q + 6)$ is positive,
$\therefore$ their product must be negative.

26. In a triangle, the sum of the lengths of any two sides must be greater than that of the third side.
 $\therefore$ $x < 6 + 3$, $x < 9$ …..(i); $6 < x + 3$, $x > 3$ …..(ii);
 $3 < 6 + x$, $x > -3$ …..(iii); $\therefore$ x must be integers from 3 to 9.
 Ans. The possible values of x are 4, 5, 6, 7 and 8.

27. $6x < -y$, $\dfrac{6x}{y} > -1$ ($\because$ $y < 0$), $\therefore$ $\dfrac{x}{y} > -\dfrac{1}{6}$

28. $k > 5$, i.e. $5 - k < 0$. $5y + k - ky \le 8 - 3k$, $(5 - k)y \le 8 - 4k$,
 $\therefore$ $y \ge \dfrac{8-4k}{5-k}$ ($\because$ $5 - k < 0$)

29. (a) $m = 1 - n$, and $m > -8$, $\therefore$ $1 - n > -8$, $-n > -9$, $n < 9$
 (b) $n = 2m + 4$, $\dfrac{n-4}{2} = m$, $\therefore$ $\dfrac{n-4}{2} > -8$, $n - 4 > -16$, $n > -12$
 (c) $2n = 7 - 3m$, $m = \dfrac{7-2n}{3}$,
 $\therefore$ $\dfrac{7-2n}{3} > -8$, $7 - 2n > -24$, $-2n > -31$, $n < \dfrac{31}{2}$

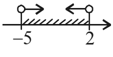
30. (a) $24 + y = 3x$, $x = \dfrac{24+y}{3}$, $\because$ $x > 0$, $\therefore$ $\dfrac{24+y}{3} > 0$, $24 + y > 0$, $y > -24$
 (b) $y = 3x - 24$, $\because$ $y \le -15$, $\therefore$ $3x - 24 \le -15$, $3x \le 9$, $x \le 3$
 (c) $\because$ $x > 7$, $\therefore$ $\dfrac{24+y}{3} > 7$, $24 + y > 21$, $y > -3$, $\therefore$ y can be 0.

31. $\dfrac{1}{x} > \dfrac{1}{y}$, $\dfrac{1}{x} - \dfrac{1}{y} > 0$, $\therefore$ $\dfrac{y-x}{xy} > 0$.
 But $x > y$, $0 > y - x$, i.e. $(y - x)$ is a negative number.
 Since $\dfrac{y-x}{xy}$ is positive but $(y - x)$ is negative, $\therefore$ xy must be negative, that is, x and y must be of opposite signs. But $x > y$, $\therefore$ $x > 0$ and $y < 0$.

32. (a) Greatest value = $3(6) + 2(-2) = 18 - 4 = 14$
 (b) The smallest value of $x^2 = 0^2 = 0$, the smallest value of $y^2 = (-2)^2 = 4$,
 $\therefore$ the smallest value $x^2 + y^2 = 0 + 4 = 4$.
 The greatest value of $x^2 = 6^2 = 36$, the greatest value of $y^2 = (-12)^2 = 144$,
 $\therefore$ the greatest value $x^2 + y^2 = 36 + 144 = 180$. $\therefore$ $4 \le x^2 + y^2 \le 180$
 (c) The smallest value of $y - x = y_{smallest} - x_{biggest} = -12 - 6 = -18$.
 The greatest value of $y - x = y_{biggest} - x_{smallest} = -2 - (-3) = 1$.
 $\therefore$ $-18 \le y - x \le 1$
 (d) Least value = $\dfrac{6}{-2} = -3$, greatest value = $\dfrac{-3}{-2} = \dfrac{3}{2}$, $\therefore$ $-3 \le \dfrac{x}{y} \le \dfrac{3}{2}$

33. (a) $12 - 3(2x + 1) \ge 4(6x - 4)$, $12 - 6x - 3 \ge 24x - 16$, $-30x \ge -25$, $\therefore$ $x \le \dfrac{5}{6}$
 (b) Let $x = \dfrac{6y+1}{3}$. The inequality becomes: $1 - \dfrac{1}{4}(2x+1) \ge \dfrac{1}{3}(6x-4)$
 From (a), $x \le \dfrac{5}{6}$, $\therefore$ $\dfrac{6y+1}{3} \le \dfrac{5}{6}$, $12y + 2 \le 5$, $y \le \dfrac{1}{4}$

34. (a) $4x + 1 > -19$, $4x > -20$, $\therefore$ $x > -5$
 (b) $4 - x < 18 - 8x$, $7x < 14$, $\therefore$ $x < 2$
 (c) $\because$ $k > -5$ and $k < 2$, $\therefore$ k are numbers from -5 to 2.

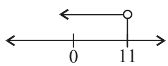


35. Let n be the number of sides. $3n \le 45$, $n \le 15$;
but the smallest number of sides is 3, $\therefore n \ge 3$.
Each interior angle = $\dfrac{(n-2)\times 180°}{n} = \dfrac{180°n - 360°}{n} = 180° - \dfrac{360°}{n}$
It is greatest when $n = 15$, and smallest when $n = 3$.
$\therefore$ The greatest interior angle $= 180° - \dfrac{360°}{15} = 156°$,
and the smallest interior angle $= 180° - \dfrac{360°}{3} = 60°$.

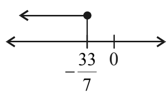
36. (a) Let x be the number of copies.
For Shop A, $1500 + 1.2x \le 2\,000$, $1.2x \le 500$, $x \le 416\frac{2}{3}$,
$\therefore$ its maximum number of copies is 416.
For Shop B, $900 + 1.5x \le 2\,000$, $1.5x \le 1\,100$, $x \le 733\frac{1}{3}$,
$\therefore$ its maximum number of copies is 733.
Ans. Print Shop B should be chosen.
(b) $1500 + 1.2x < 900 + 1.5x$, $-0.3x < -600$, $\therefore x > 2\,000$
Ans. It is cheaper to choose A when the number of copies is more than 2 000.

37. (a) Let x be the no. of \$2 coins. $\therefore$ no. of \$5 coins = $\dfrac{140-2x}{5} = 28 - \dfrac{2}{5}x$.
$\because$ the no. of coins must be an integer, $\therefore \dfrac{2}{5}x$ must be an integer,
$\therefore x$ must be a multiple of 5, i.e. the no. of \$2 coins must be a multiple of 5.
(b) $\dfrac{140-2x}{5} - x > 3$, $140 - 2x - 5x \ge 15$, $-7x \ge -125$, $x \le 17.9$;
$\because x$ must be a multiple of 5, $\therefore x = 15$.
Ans. The maximum number of \$2 coins is 15.

38. $y = 2 - \dfrac{3}{4}(x-5)$, $4y = 8 - 3x + 15$, $x = \dfrac{23-4y}{3}$.
$\therefore \dfrac{23-4y}{3} > -7$, $23 - 4y > -21$,
$-4y > -44$, $y < 11$.

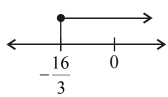


39. (a) $2 - \dfrac{3x+7}{5} \ge \dfrac{9-x}{4}$; $40 - 4(3x+7) \ge 5(9-x)$;
$40 - 12x - 28 \ge 45 - 5x$; $-33 \ge 7x$; $x \le -\dfrac{33}{7}$.



(b) −5, −6, −7, −8, −9, −10.

40. (a) $4\left(1 + \dfrac{x-3}{5}\right) \ge \dfrac{x}{2}$, $4\left(\dfrac{2+x}{5}\right) \ge \dfrac{x}{2}$,
$8(2+x) \ge 5x$, $16 + 8x \ge 5x$,
$3x \ge -16$, $x \ge -\dfrac{16}{3}$.



(b) $4\left(1-\frac{6y+3}{5}\right) \geq -3y$, $4\left(1+\frac{-6y-3}{5}\right) \geq \frac{-6y}{2}$

From (a), $-6y \geq -\frac{16}{3}$, $y = \frac{8}{9}$.

41. (a) $4 - x < \frac{3x+8}{5}$, $5(4 - x) < 3x + 8$,

$20 - 5x < 3x + 8$, $12 < 8x$, $x > \frac{3}{2}$.

(b) $4 - y + k < \frac{3y - 3k + 8}{5}$, $\therefore 4 - (y - k) < \frac{3(y-k)+8}{5}$

From (a), $y - k > \frac{3}{2}$, i.e. $y - \frac{3}{2} > k$

$\because k > 1$, $\therefore y - \frac{3}{2} > 1$, $y > \frac{5}{2}$,

$\therefore$ The least integral value of y is 3.

Unit 4 Percentages: Interests, growth & decay

1. Simple interest = \$8 800 × 4% × 3.5 = \$1 232. Amount = \$(8 800 + \$1 232) = \$10 032
2. Let \$$P$ be the principal. $P \times 5\% \times \frac{9}{12} = 360$, $P = 9\,600$. *Ans. The principal is \$9 600.*
3. Let n years be the time taken. $6\,000 \times 3\% \times n = 4050$, $n = 2.25$.
   Ans. It takes 2.25 years to earn \$4 050 simple interest.
4. Let r% p.a. be the interest rate. $72\,000 \times r\% \times \frac{1}{4} = 1\,080$, $r\% = 0.06 = 6\%$.
   Ans. The interest rate is 6% p.a.
5. Let \$$P$ be the principal. $P \times (5\% - 3\%) = 230$, $P \times 2\% = 230$, $P = 11\,500$.
   Ans. His principal is \$11 500.
6. Amount =\$65 000 × [(1 + 4% × 2 + 5% × (5.5 – 2)] = \$81 575.
7. (a) Amount $= \$50\,000(1+\frac{6}{2}\%)^{2\times2} = \$56\,275$

   (b) Amount $= \$50\,000(1+\frac{6}{4}\%)^{2\times4} = \$56\,325$

   (c) Amount $= \$50\,000(1+\frac{6}{12}\%)^{2\times12} = \$56\,358$

   (d) Amount $= \$50\,000(1+\frac{6}{365}\%)^{2\times365} = \$56\,374$
8. (a) Amount $= \$64\,000(1+\frac{12}{12}\%)^{3\frac{3}{4}\times12} = \$64\,000(1+1\%)^{45} = \$100\,148$

   (b) Total interest $= \$100\,148 - \$64\,000 = \$36\,148$
9. (a) Interest $= \$280\,000[(1+10\%)^3 - 1] = \$92\,680$

   (b) Interest $= \$280\,000[(1+\frac{10}{12}\%)^{3\times12} - 1] = \$97\,491$

10. Compound interest $= \$55\,000(1+5\%)^4 - \$55\,000 = \$11\,853$,
Simple interest $= \$55\,000 \times 5\% \times 4 = \$11\,000$, $\therefore$ Difference $= \$11\,853 - \$11\,000 = \$853$

11. (a) Amount $= \$28\,000(1+\frac{4}{4}\%)^{1\times4}(1+\frac{8}{4}\%)^{2.5\times4} = \$28\,000(1.01)^4(1.02)^{10}$ $=\$35\,518$

(b) Total interest $= \$35\,518 - \$28\,000 = \$7\,518$

12. (a) Amount $= \$15\,000\ (1 + 7\%)^5 + \$15\,000\ (1 + 7\%)^4 + \$15\,000\ (1 + 7\%)^3 +$
$\$15\,000\ (1 + 7\%)^2 + \$15\,000\ (1 + 7\%)$
$= \$15\,000\ (1.07^5 + 1.07^4 + 1.07^3 + 1.07^2 + 1.07) = \$92\,299$

(b) Total interest $= \$92\,299 - \$15\,000 \times 5 = \$17\,299$

13. (a) Let \$P be the principal. $P(1+\frac{20}{2}\%)^{1.5\times2} = 114\,466$, $P(1.1)^3 = 114\,466$,
$P = 86\,000$. *Ans. The principal is \$86 000.*

(b) Total interest $= \$114{,}466 - \$86{,}000 = \$28{,}466$

14. After 1st payment, amount he owes $= \$20\,000(1+10\%) - \$5\,000 = \$17\,000$;
After 2nd payment, amount he owes $= \$17\,000(1+10\%) - \$5\,000 = \$13\,700$;
After 3rd payment, amount he owes $= \$13\,700(1+10\%) - \$5\,000 = \$10\,070$;
$\therefore$ After 4th payment, amount he owes $= \$10\,070(1+10\%) - \$5\,000 = \$6\,077$.

15. (a) Growth factor $= 1 + 15\% = 1.15$

(b) Increase in members $= 3\,000(1.15)^4 - 3\,000 = 3\,000[(1.15)^4 - 1] = 2\,247$

16. Decay factor $= \frac{4\,000}{4\,500} = \frac{8}{9}$

17. Value $= \$7\,500(1-9\%)^{3\times2} = \$7\,500(0.91)^6 = \$4\,259$

18. (a) Salary after 3 years $= \$24\,200(1+10\%)^3 = \$32\,210$

(b) Let \$$y$ be the salary 2 years ago. $y(1+10\%)^2 = 24\,200$, $y = 20\,000$.
Ans. Her salary was \$20 000 2 years ago.

19. (a) Volume after one day $= 3\,000(1-6\%)^{24} = 680cm^3$

(b) Let y cm^3 be the volume 3 hours ago. $y(1-6\%)^3 = 3\,000$, $y = 3\,612$.
Ans. The volume of the balloon was 3 612 cm^3 *3 hours ago.*

20. (a) Value after 1 year $= \$3100(1+2\%)^3 \approx \$3\,289.7 \approx \$3\,290$

(b) Value after 2 years $= \$3\,289.7(1-2\%)^4 = \$3\,034$

21. (a) At beginning of 2nd month, amount he owed $= \$35\,000(1+\frac{18}{12}\%) - \$10\,000 = \$25\,525$
$\therefore$ At beginning of 3rd month, amount he owed $= \$25\,525(1+1.5\%) - \$10\,000 = \$15\,908$

(b) At beginning of 4th month, amount he owed $= \$15\,908(1+1.5\%) - \$10\,000$
$= \$6\,146$, at end of 4th month, amount he owed $= \$6\,146(1+1.5\%)$
$= \$6\,239 < \$10\,000$. *Ans. 4 payments were needed.*

22. Amount owed after 1st installment $= \$8\,000 - \$1\,200 = \$6\,800$
Amount he owed after 2nd installment $= \$6\,800(1+\frac{10}{12}\%) - \$900 = \$5\,956.67$
$\therefore$ Amount he owed after 3rd installment $= \$5\,956.67(1+\frac{10}{12}\%) - \$900 = \$5\,106$

23. Amount at end of 1st year $= \$20\,000(1+\frac{5}{2}\%)^2$

Amount at end of 2nd year $= \$20\,000(1+\frac{5}{2}\%)^4$

$\therefore$ Interest earned in 2nd year $= \$20\,000\ [(1.025)^4-(1.025)^2] = \$1\,064$

24. Let x be the growth factor. $18x^2 = 36, x^2 = 2, x = \sqrt{2}$

$\therefore$ No. of bacteria in $\frac{1}{2}$ hour $= 18(\sqrt{2})^{30} = 590\,000$ (corr. to nearest 1 000)

25. Let $\$x$ be the amount of each payment. $[90\,000(1+25\%)-x](1+25\%)-x=0,$

$(112\,500-x)(1.25)-x=0, \quad 140\,625 - 2.25x = 0, \quad x = 62\,500,$

$\therefore$ Total interest $= \$62\,500\times 2 - \$90\,000 = \$35\,000$

26. Let $\$P$ and $r\%$ be the principal and the minimum interest rate respectively.

$P(1+r\%)^{10} \geq P(1+100\%), (1+r\%)^{10} \geq 2, \quad 1 + r\% \geq 1.0718, \quad r\% \geq 7.18\%.$

Ans. The minimum interest rate is 7.18%.

27. (a) Value of his flat $= \$4\,000\,000(1+10\%)^3 = \$5\,324\,000$

(b) Amount he owes Peter $= \$2\,500\,000(1+\frac{8}{12}\%)^{3\times 12} = \$3\,175\,593$

(c) Increase in value of the flat $= \$(5\,324\,000 - 4\,000\,000) = \$1\,324\,000$

Interest he has to pay to Peter = $\$(3\,175\,593 - 2\,500\,000) = \$675\,593$

The profit = $\$(1\,324\,000 - 675\,593 - 380\,000) = \$268\,407$

28. (a) His debt $= \$30\,000(1+40\%)^3 = \$82\,320$

(b) After 1st payment $= \$30\,000(1+40\%) - \$15\,000 = \$27\,000$,

After 2nd payment $= \$27\,000(1+40\%) - \$15\,000 = \$22\,800$,

$\therefore$ Amount owed after 3rd payment $= \$22\,800(1+40\%) - \$15\,000 = \$16\,920$

(c) Amount he owes after 1month $= \$30\,000(1+40\%) - \$10\,000 = \$32\,000$,

$\because$ The amount keeps increasing, $\therefore$ he can never clear his debt.

(d) Amount he owes after 1month $= \$30\,000(1+40\%) - \$12\,000 = \$30\,000$

= the principal, $\therefore$ He will still owe the loan shark \$30 000 after 20 years.

29. (a) Interest received by the bank = $24\,000\left(1+\frac{8\%}{2}\right)^2 - 24\,000 = \$1\,958.4$

(b) The amount he owes the bank after the first repayment

$= 24\,000\left(1+\frac{8\%}{2}\right) - 8\,000 = \$16\,960$

Total interest received by the bank

$= [16\,960\left(1+\frac{8\%}{2}\right) + 8\,000] - 24\,000 = 25\,638.4 - 24\,000 = \$1\,638.4$

30. (a) Interest received if depositing in Bank A

$= 36\,000\left(1+\frac{4.4\%}{4}\right)^{4\times 4} - 36\,000 = 42\,886.54 - 36\,000 = \$6\,886.54$ (2 d.p.)

(b) Interest received if depositing in Bank B

$= 36\,000\left(1+\frac{4.2\%}{12}\right)^{4\times 12} - 36\,000 = 42\,573.23 - 36\,000 = \$6\,573.23$ (2 d.p.)

$\because$ \$6 886.54 > \$6 573.23 , $\therefore$ Sally should deposit the money in Bank A.

31. (a) Let r% be the growth rate.

$2(1 + r\%)^3 = 2.662$, $(1 + r\%)^3 = 1.331$, $1 + r\% = 1.1$, $r =\% = 0.1$, $r = 10$.

$\therefore$ Growth rate is 10%.

(b) Population of the city in 2024 = 2.662 × 1.1 million = 2.9282 million < 3 million

Population of the city in 2025 = 2.662×1.1^2 million = 3.22102 million > 3 million

$\therefore$ In 2025.

(c) Population of the city in 2027

$= 2.662 \times (1 + 10\%)^4$ million = 3.90 million (3 sig. fig.)

32. (a) $600\,000\,(5\% \times 2) = \left[600\,000\left(1+\frac{r\%}{12}\right)^{12} - 600\,000\right](1 - 12\%)$

$0.1 = \left[\left(1+\frac{r\%}{12}\right)^{12} - 1\right](0.88)$, $\left(1+\frac{r\%}{12}\right)^{12} = \frac{49}{44}$,

$r\% = \left(\sqrt[12]{\frac{49}{44}} - 1\right)\times 12$, $r\% = 0.108$ (3 sig. fig.) , $r = 10.8$

(b) $\because$ 10 < 10.8 , $\therefore$ He will choose Option I to get more interest.

Unit 5 Multiple percentage changes & salaries tax

1. The amount he spent = $1\,440 \div 15\% = \$9\,600$
2. His original budget = $88\,000 \div (1+20\%) = \$73\,300$ (3 sig. fig.)
3. His present monthly income = $15\,000\,(1-30\%)\,(1+20\%) = \$12\,600$
4. Let the original length and width be x and y respectively.

   Original area = xy. New area = $x(1-10\%)\times y(1-12\%) = 0.792xy$

   Percentage decrease in area = $\frac{xy - 0.792xy}{xy}\times 100\% = (1-0.792)\times 100\% = 20.8\%$
5. Let the side of a small cube be x, $\therefore$ the side of the large cube $= \sqrt[3]{8x^3} = 2x$.

   Percentage increase in total surface area = $\frac{8\times 6x^2 - 6(2x)^2}{6(2x)^2}\times 100\% = \frac{48-24}{24}\times 100\% = 100\%$
6. Her weight before diet = $90 \div (1+25\%) \div (1-10\%) = 80$kg
7. Let the original base and height be x and y respectively.

   Original area $= \frac{1}{2}xy$. New area = $\frac{1}{2}\times x(1+28\%)\times y(1-12\%) = 0.5632xy$

   Percentage change in area = $\frac{0.5632xy - 0.5xy}{0.5xy}\times 100\% = \frac{0.0632}{0.5}\times 100\% = 12.64\%$ (increase)
8. Length of the fish two months ago = $40 \div (1+15\%)^2 = 30.2$cm (3 sig. fig.)
9. Total pass percentage = $60\% + (1-60\%)(1-45\%) = 60\% + 22\% = 82\%$

10. Amount paid by Miss Ng = $60\,000\,(1+5\%)\,(1-8\%) = \$57\,960$
11. The cost paid by Andrew = $282 \div (1+50\%) \div (1-6\%) = \$200$
12. Percentage change in the running cost
 $= [45\%\,(1+20\%) + 35\%\,(1-5\%) + 20\%\,(1+10\%)] - 100\%$
 $= 54\% + 33.25\% + 22\% - 100\% = 9.25\%$ (increase)
13. $A = C\,(1-10\%)\,(1+20\%) = 1.08\,C$, $\therefore$ $A : C = 1.08 : 1 = 108 : 100 = 27 : 25$
14. (a) Let x cm be the height of Helen.
 Height of May = $x(1-15\%)\,(1+10\%) = 0.935x$
 Percentage difference = $\dfrac{0.935x - x}{x} \times 100\% = -0.065 \times 100\% = -6.5\%$
 Ans. May is shorter than Helen by 6.5%.
 (b) Percentage difference = $\dfrac{x - 0.935x}{0.935x} \times 100\% = 6.95\%$
 Ans. Helen is taller than May by about 6.95%.
15. (a) $\because$ Net chargeable income = 128 000 – 132 000 < \$0, $\therefore$ salaries tax = \$0.
 (b) Net chargeable income = 294 000 – 132 000 = 162 000 = \$50 000 × 3 + \$12 000
 Salaries tax = 50 000 × (2% + 6% + 10%) + \$12 000 × 14% = 9 000 + 1 680 = \$10 680
 (c) Net chargeable income = 272 000 – 182 000 = 90 000 = \$50 000 + \$40 000
 Salaries tax = 50 000 × 2% + 40 000 × 6% = \$3 400
 (d) Net chargeable income = 534 000 – 282 000 = 252 000 = \$50 000 × 4 + \$52 000
 Salaries tax = 50 000 × (2% + 6% + 10% + 14%) + 52 000 × 17%
 = 16 000 + 8 840 = \$24 840
16. Net chargeable income = 26 500 × 12 – 132 000 = 86 000 = \$50 000 + \$46 000
 Salaries tax = 50 000 × 2% + 36 000 × 6% = \$3 160
17. Max. total income = total allowance = \$232 000
18. Salaries tax on the first \$200 000 = 50 000 × (2% + 6% + 10% + 14%) = \$16 000
 $\therefore$ Annual income = 200 000 + (17 020 – 16 000)÷ 17% + 132 000 = \$338 000
19. Net chargeable income = 4 200 000 – 332 000 = 3 868 000 = \$50 000 × 4 + \$3 668 000
 Progressive salaries tax = 50 000 × (2% + 6% + 10% + 14%) + 3 668 000 × 17%
 = 16 000 + 623 560 = \$639 560
 Upper limit of salaries tax = 4 200 000 × 15% = \$630 000 < \$639 560
 $\therefore$ salaries tax = \$630 000
20. Let his annual income be \$$x$. The standard rate is 15%.
 $x \times 15\% = 50\,000 \times (2\% + 6\% + 10\% + 14\%) + (x - 282\,000) \times 17\%$,
 $16\,000 + (x - 282\,000) \times 0.17 = 0.15x$, $0.02x = \$31\,940$, $x = 1\,597\,000$
 Ans. His annual income is \$1 597 000.
21. Let A be the original area, r_1 be the original radius and r_2 be the new radius,
 $\pi r_1^2 = A$ ……(1) and $\pi r_2^2 = A\,(1-11.64\%) = 0.8836A$ ……(2)
 $\dfrac{(2)}{(1)} : 0.8836 = \dfrac{r_2^2}{r_1^2}$, $\sqrt{r_2^2} = \sqrt{0.8836 r_1^2}$, $r_2 = 0.94 r_1$

Percentage decrease in radius = $\dfrac{(r_1 - 0.94r_1)}{r_1} \times 100\% = 6\%$

22. Let $n\%$ be the percentage change in height.

$15\ (1 + 20\%) \times 18\ (1 + 15\%) \times 18\ (1 + n\%) = 15 \times 8 \times 18 \times (1 - 10\%)$,

$n\% = \dfrac{0.9}{(1.2)(1.15)} - 1$, $n = -34.8$ *Ans.* *The percentage decrease in height is 34.8%*

23. Let the length be ℓ cm and the width be w cm.

$\ell \times (1+12\%) = w \times (1-20\%)$, $\ell = \dfrac{80}{112} w = \dfrac{5}{7} w$. The percentage = $(1 - \dfrac{5}{7}) \times 100\% = 28.6\%$

Ans. *The length is shorter than the width by 28.6%.*

24. Let the percentage decrease be $r\%$, and the original weight be W kg.

$W\ (1+20\%)\ (1-r\%) = W$, $1 - r\% = \dfrac{100}{120} = \dfrac{5}{6}$, $r\% = \dfrac{1}{6} = \dfrac{1}{6} \times 100\% = 16\dfrac{2}{3}\%$

Ans. *The percentage decrease should be* $16\dfrac{2}{3}\%$.

25. Let the number be N. $N(1 + x\%)\ (1 - y\%) = N$, $(1 + x\%)\ (\dfrac{100 - y}{100}) = 1$,

$x\% = \dfrac{100}{100 - y} - 1 = \dfrac{100 - (100 - y)}{100 - y} = \dfrac{y}{100 - y}$, $\therefore$ $x = \dfrac{100y}{100 - y}$

26. Let his original hourly income be $\$x$.

$\therefore$ Percentage change = $\dfrac{9x \times (1 - 15\%) - 8x}{8x} \times 100\% = -4.375\%$ (decrease)

27. Let the original total cost be $\$C$.

	Material A	Material B	Material C
Original cost	$C \times \dfrac{1}{1+5+4} = 0.1C$	$C \times \dfrac{5}{1+5+4} = 0.5C$	$C \times \dfrac{4}{1+5+4} = 0.4C$
New cost	$0.1C \times (1+30\%) = 0.13C$	$0.5C \times (1+2\%) = 0.51C$	$0.4C \times (1+25\%) = 0.5C$

New total cost = $(0.13 + 0.51 + 0.5)C = 1.14C$

Overall percentage increase = $(1.14 - 1) \times 100\% = 14\%$.

28. Percentage change = $(1+15\%)\ (1-15\%) - 100\% = -2.25\%$ (decrease)

29. Let the selling price of one pen be $\$S$ and the cost price be \$C.

$15S = 18C$, $S = 1.2C$. Profit percent = $\dfrac{S - C}{C} \times 100\% = \dfrac{1.2C - C}{C} \times 100\% = 20\%$.

30. (a) Profit percentage = $(1+40\%)\ (1-25\%) - 100\% = 5\%$

(b) Marked price = $8\,820 \div (1 - 25\%) = \$11\,760$. Cost price = $11\,760 \div (1 + 40\%) = \$8\,400$

31. The cost paid by B = $2\,100 \div 15\% = \$14\,000$.

$\therefore$ The amount paid by C = $14\,000 + 2\,100 = \$16\,100$,

and profit gained by A = $14\,000 - 14\,000 \div (1 + 28\%) = \$3\,062.5$

32. Total cost price = $3\,300 \div (1+10\%) + 2\,100 \div (1-20\%) = 3\,000 + 2\,625 = \$5\,625$

Selling price of wardrobe = $5\,625 - 2\,500 = \$3\,125$

33. The total cost price of the watches = $x \div (1+15\%) + x \div (1-20\%)$ $= 2.1196x$ (5 sig. fig.)

    $\therefore$ The profit percentage = $\dfrac{2.1196x - 2x}{2.1196x} \times 100\%$ $= 5.64\%$

    Ans. The percentage profit on the whole is 5.64%

34. Let the total number of monitors be N, and the cost price of each monitor be \$$y$,

    $50y(1+40\%) + (N-50)\,y(1-40\%) = Ny(1+10\%)$,

    $70 + 0.6N - 30 = 1.1N$, $40 = 0.5N$, $N = 80$

    Ans. The merchant had 80 monitors at the beginning.

35. (a) Let the number be N, $N(1-10\%)(1+50\%) = 810$, $1.35N = 810$, $N = 600$

    Ans. The original number is 600.

    (b) $600 \times (1+25\%) \times (1-y\%) = 600 + 75$, $1 - y\% = \dfrac{675}{600 \times 1.25} = 0.9$,

    $y\% = 0.1 = 10\%$, $\therefore$ $y = 10$

36. Let P kg be Paul 's weight. $D = P(1+20\%)$, $D = \dfrac{120}{100}P$, $\therefore$ $P = \dfrac{5}{6}D$

    Weight of Mr. Lau = $D + \dfrac{5}{6}D = \dfrac{11}{6}D = D \times \dfrac{11}{6} \times 100\% = D \times 183\dfrac{1}{3}\%$.

37. Let his monthly income be \$$I$. Original expenditure = $I(1-30\%) = 0.7I$;

    new expenditure = new income − new savings

    $= I(1+5\%) - I \times 30\% \times (1+20\%)$ $= 1.05I - 0.36I = 0.69I$,

    Percentage change in his monthly expenditure $= \dfrac{0.69I - 0.7I}{0.7I} \times 100\%$ $= -1.43\%$ (3 sig. fig.)

38. Let his original income be \$$I$. New expenditure = new income × (1 − 20%),

    $\therefore$ new income = $I(1-10\%)(1+20\%) \div (1-20\%) = 1.35I$.

    $\therefore$ Percentage increase in his income = $(1.35 - 1) \times 100\% = 35\%$

39. Ratio of liquid A to liquid B in the mixture = $1\,000 : 250 = 4 : 1$.

    Amount of liquid B remained = $(1\,000 + 250) \times (1-40\%) \times \dfrac{1}{4+1} = 150$ cm^3.

40. Let y cm^3 be the amount of water to be added.

    Amount of alcohol = 1 000 × 60% = (1000 + y) × 20%,

    1 000 + y = 3000, y = 2 000. *Ans. 2 000* cm^3 *of water should be added.*

41. Let the amount of salt to be added be k kg.

    $x \cdot x\% = (x+k) \cdot \dfrac{x}{2}\%$, $x = (x+k) \cdot \dfrac{1}{2}$, $2x = x + k$, $\therefore$ $k = x$

    Ans. x kg of water should be added to the salt water.

42. Let t_1, v_1 and v_2, and be the original walking time, original speed, and the new speed respectively. The distance = $v_1 t_1 = v_2 t_1(1-20\%)$, $v_1 = 0.8v_2$, $v_2 = 1.25v_1$,

    $\therefore$ The percentage change in speed = $(1.25 - 1) \times 100\% = 25\%$

    Ans. Miss Chan should increase her walking speed by 25%.

43. Let his original speed be v_1 m/min, new speed be v_2 m/min.

$45v_1 = (45-5)v_2,\quad v_2 = \frac{9}{8}v_1$

$\therefore$ Percentage increase in his speed = $(\frac{9}{8}-1)\times 100\% = 12.5\%$

44. Let the distance between P and Q be d m, and his speed in the return trip be x km/h.

The total time take = $\frac{d}{60}+\frac{d}{x}=\frac{2d}{72}$, $\therefore$ $\frac{1}{60}+\frac{1}{x}=\frac{1}{36}$, $\frac{1}{x}=\frac{1}{36}-\frac{1}{60}=\frac{1}{90}$, $\therefore$ $x = 90$.

Percentage increase in speed $= \frac{90-60}{60} = \times 100\% = 50\%$

45. (a) Let x kg be Paul's original weight.

Overall percentage change $= \frac{(1-6\%)(1-5\%)x-x}{x}\times 100\% = -10.7\%$

(b) (i) $x(10.7\%) = 9.63, x = \frac{9.63}{10.7\%} = 90$, $\therefore$ Paul's original weight is 90 kg.

(ii) Required percentage increase $= \frac{9.63}{90-9.63}\times 100\% = 12.0\%$ (3 sig. fig.)

46. (a) Required height $= 40 \div (1+3\%)\div(1+5\%) = 37.0$ cm (cor. to 3 sig. fig.)

(b) Required height $= 40(1+8\%)(1+4\%) = 44.928$ cm $= 44.9$ cm (3 sig. fig.)

(c) The percentage change $= \frac{44.928-37.0}{37.0}\times 100\% = 21.4\% \neq 20\%$

Thus, the claim is disagreed.

47. (a) The number of laptops sold this month = $25(1-12\%) = 22$

The required percentage change

$= \frac{(20+22)C-(20+25)C}{(20+25)C}\times 100\% = \frac{-3C}{45C}\times 100\% = -6\frac{2}{3}\%$

(b) Monthly income last month $= 1\,500 \div 6\frac{2}{3}\% = \$22\,500$

Monthly income this month = 22 500 – 1 500 = $21 000

(c) The new commission = $\$500(1-16\%) = \$420$

Let x be required number of laptops sold next month.

$(20+x)(420) = 21\,000,\quad 20+x = 50,\quad x = 30$

The required percentage change $= \frac{30-22}{22}\times 100\% = 36.4\%$

Unit 6 Quadrilaterals

In questions (1 – 4), the following reasons are simplified:	*(1) the properties of parallelogram as "//gram";* *(2) ∠sum of Δ as "Δ";* *(3) adj. ∠s on st. line as "st. line".*

1. (a) $a+4 = 20-a$ (// gram), $2a = 16$, $\therefore$ $a = 8°$

$x + 50° = 3x - 70°$ (// gram), $120° = 2x$, $\therefore$ $x = 60°$

$y+x+50°=180°$ (int. $\angle$s, AB // DC), $\therefore$ $y=180°-60°-50°=70°$

(b) $3x+5=5x-3$ (// gram), $8=2x$, $\therefore$ $x=4$.
$7x-y=7y+x$ (// gram), $7(4)-y=7y+4$, $\therefore$ $y=3$

(c) $m+155°=180°$ (int.$\angle$s, QP // RS), $\therefore$ $m=180°-155°=25°$
$n+2m=180°$ (int.$\angle$s, QP // PS), $\therefore$ $n=180°-2(25°)=130°$

(d) $\angle$PSR = c (// gram), $\angle$PSR + 70° =180° (st. line), $\therefore$ $c=180°-70°=110°$.
$c+d+10°+2d=180°$ (Δ), $3d=180°-110°-10°=60°$, $\therefore$ $d=20°$

(e) $\angle$DFE $=180°-102°=78°$ (st. line), $\angle$DEF = $\angle$DFE = 78° (base $\angle$s, isos.Δ),
$\angle$D = 180° – 2(78°) = 24° (Δ), $2x$ = 24° (// gram), $\therefore$ x = 12°
$y-x+24°=180°$ (int.$\angle$s, CD//BA), $\therefore$ $y=180°+12°-24°=168°$

2. (a) $3x=90°$ (rectangle), $\therefore$ $x=30°$. $2y-x+2x+y+90°=180°$ (Δ),
$3y+x=90°$, $3y=90°-30°=60°$, $\therefore$ $y=20°$

(b) $\angle$SRT = 90° – 65° = 25°, $\therefore$ $c=\angle$SRT = 25° (base$\angle$s, isos.Δ)
$e=180°-25°-25°=130°$ (Δ), $d=90°-c=90°-25°=65°$

(c) $\angle$PST = y (rectangle), $y+\angle$PST = 100° (ext.$\angle$of Δ), $2y$ = 100°,
$\therefore$ y = 50°. $x+\angle$PST + 90° = 180° (Δ), $\therefore$ $x=180°-90°-50°=40°$

(d) RS = 12 (rectangle), $12^2+(a-4)^2=(a+4)^2$ (Pyth. Thm.),
$144+a^2-8a+16=a^2+8a+16$, $144=16a$, $\therefore$ $a=9$

(e) $a+90°=4a+15°$ (ext.$\angle$of Δ), $75°=3a$, $\therefore$ $a=25^o$
$\angle$APQ = a = 25° (base$\angle$s, isos.Δ), $\angle$QAP = $180°-25°-25°=130°$ (Δ),
$\therefore$ $b=\angle$QAP = 130° (vert. opp.$\angle$s)

3. (a) $3m+8=13-2m$ (rhombus), $5m=5$, $\therefore$ $m=1$. $6a=144°$ (rhombus),
$\therefore$ a = 24°. $4b+144°=180°$ (int.$\angle$s, BA//CD), 4b = 36°, $\therefore$ b = 9°

(b) $4x$ = 24° (rhombus), $\therefore$ x = 6°. $4x+y$ + 90° = 180° (Δ),
$\therefore$ $y=180°-90°-24°=66°$. $5z$ = 90° (rhombus), $\therefore$ z =18°

(c) $3x$ = 51° (alt.$\angle$s, SR // PQ), $\therefore$ x = 17°. y + 11° + 2(3x) = 180° (Δ),
$\therefore$ $y=180°-11°-2(51°)=67°$

(d) $7m-6=5m-2$ (rhombus), $2m=4$, $\therefore$ $m=2$. HG = 6 (rhombus),
HD = 7(2) – 6 = 8, $\therefore$ $n=\sqrt{6^2+8^2}=\sqrt{100}=10$ (Pyth. Thm.)

4. (a) $3x=16-x$ (square), $4x=16$, $\therefore$ $x=4$.
$y+2y=45°$ (square), $3y=45°$, $\therefore$ $y=15°$

(b) $2a+14°=90°$ (square), $2a=76°$, $\therefore$ $a=38°$. $8b=4$ (square), $\therefore$ $b=\dfrac{1}{2}$

(c) $\angle$ACB = 45° (square), $m+45°=110°$ (ext.$\angle$ of Δ), $\therefore$ $m=65°$;
$m+n=180°$ (int.$\angle$ s, AD//BC), $\therefore$ $n=180°-65°=115°$

(d) $4y-3°=45°$ (rhombus), $4y=48°$, $\therefore$ $y=12°$. QT = TR = 5 (square),
$x^2=5^2+5^2$ (Pyth. Thm.), $x^2=50$, $\therefore$ $x=\sqrt{50}=5\sqrt{2}$

5. In ΔPSX and ΔRQY, PS = RQ (prop. of // gram), $\angle$P = $\angle$R (prop. of // gram),
PX = RY (given), $\therefore$ ΔPSX $\cong$ ΔRQY (S.A.S.), $\therefore$ QY = XS (corr. sides, $\cong$ Δs)

6. $\because$ AB = DC and DC = EF (prop. of // gram), $\therefore$ AB = EF;
$\because$ AB // DC and DC // EF (def. of //gram), $\therefore$ AB // EF,
$\therefore$ ABFE is a // gram (2 sides eq. and //)

7. (a) In ΔABE and ΔCDF, AB = CD (prop. of // gram),
$\angle$BAE = $\angle$DCF (alt. $\angle$s, AB // DC), $\because$ AF = CE (given),
$\therefore$ AE = AF + FE = CE + FE = CF, $\therefore$ $\Delta ABE \cong \Delta CDF$ (S.A.S.)
(b) $\because$ $\Delta ABE \cong \Delta CDF$ (proved), $\therefore$ BE = DF (corr. sides, $\cong \Delta$s),
$\angle$BEA = $\angle$DFC (corr. $\angle$s, $\cong \Delta$s), $\therefore$ BE // FD (alt.$\angle$ s, eq.),
$\therefore$ BEDF is a //gram (2 sides eq. and //)

8. (a) In ΔPQN and ΔRSM, PQ = RS (prop. of //gram), $\angle$PNQ = $\angle$RMS = 90° (given),
$\angle$QPN = $\angle$SRM (alt. $\angle$s, PQ//SR), $\therefore$ $\Delta PQN \cong \Delta RSM$ (AAS)
(b) $\because$ $\Delta PQN \cong \Delta RSM$ (proved), $\therefore$ QN = SM (corr. sides, $\cong\Delta$s),
$\because \angle$PNQ = $\angle$RMS = 90° (given), $\therefore$ QN//SM (alt. $\angle$s, eq.),
$\therefore$ QNSM is a // gram (2 sides eq. and //)

9. $\angle FEG = \angle FGE = (180° - 30°) \div 2 = 75°$ (base $\angle$s, isos.Δ)
$\angle$CFG + 30° = 90° (square), $\angle$CFG = 90° – 30° = 60°.
$\because$ CF = FE = FG (square), $\therefore$ $\angle$FCG = $\angle$FGC = (180° – 60°) ÷ 2 = 60° (base $\angle$s, isos.Δ),
$\therefore$ ΔCFG is equilateral, and FG = CF = CG = CD,
$\angle$DCG = 90° – 60° = 30°. In ΔCDG and ΔFEG,
$\angle$DCG = $\angle$EFG = 30° (proved), CD = FE (square), CG = FG (proved),
$\therefore$ $\Delta CDG \cong \Delta FEG$ (S.A.S.), $\therefore$ $m = \angle CDG = 75°$ (corr. $\angle$s, $\cong\Delta$s).
$n + \angle CDG = 90°$ (square), $\therefore$ $n = 90° - 75° = 15°$

10. $\angle$ADE = 60° and AD = DE (equilateral Δ), $\angle$ADC = 90° and AD = DC,
$\therefore \angle CDE = 90° - 60° = 30°$ and DE = DC, $\therefore$ $\angle DEC = x$ (base $\angle$s, isos. Δs),
$\therefore$ $2x + 30° = 180°$ ($\angle$sum of Δ), $2x = 150°$, $\therefore$ $x = \angle DEC = 75^o$
Similarly, $\angle$AEB = 75°, but $\angle$AED = 60°,
$\therefore$ $y + 75° + 60° + 75° = 360°$ ($\angle$s at a pt.), $\therefore$ $y = 150°$

11. $\angle ADC = 180° - 115° = 65°$ (adj. $\angle$s on st. line),
$\therefore$ $m = \angle ADC = 65°$ (prop. of // gram), $\angle DCG = m = 65°$ (corr. $\angle$s, AB // DC),
$\therefore$ $n = \angle DCG = 65°$ (prop. of // gram). $\because$ EG = FG (given),
$\therefore$ $\angle FEG = n = 65°$ (base $\angle$s, isos.Δ), but $\angle$FED = 115° (alt. $\angle$s, EF // AD),
$\therefore$ $p = 115° - 65° = 50°$

12. (a) In ΔCMB and ΔCMN, CM = CM (common), $\angle$BCM = $\angle$NCM (given),
$\angle$B = $\angle$CNM = 90°, $\therefore$ $\Delta CMB \cong \Delta CMN$ (AAS)
(b) $\angle$NAM = 45° (prop of square),
$\angle NMA = 180° - 90° - 45° = 45°$ ($\angle$ sum of Δ),
$\therefore$ AN = MN (sides opp. eq. $\angle$s), but MN = MB (corr. sides $\cong\Delta$s), $\therefore$ AN = MB

13. PS = QR = 10cm (prop. of //gram), PT = 10 – 3 = 7= PQ
$\therefore \angle$PQT = $\angle$PTQ (base $\angle$s, isos.Δ), but $\angle$PTQ = $\angle$RQT (alt. $\angle$s, QR // PS),
$\therefore$ $\angle$PQT = $\angle$RQT, i.e. QT bisects $\angle$PQR.

14. (a) $\angle BCM = 180° - 90° - \angle CBN = 90° - \angle CBN$ ($\angle$ sum of Δ),
$\angle ABN + \angle CBN = 90°$ (prop. of square), $\therefore$ $\angle ABN = 90° - \angle CBN = \angle BCM$
(b) In ΔCBM and ΔBAN, $\angle BCM = \angle ABN$ (proved),
$\angle MBC = \angle A = 90°$ (prop. of square), BC = AB (prop. of square),
$\therefore$ $\Delta CBM \cong \Delta BAN$ (A.S.A.), $\therefore$ BN = CM (corr. sides, $\cong\Delta$s)
15. In ΔEP and ΔEQ, BE = DE (prop. of // gram),
$\angle PBE$ = QDE and $\angle BPE = \angle DQE$ (alt.$\angle$ s, AB // DC), $\therefore$ $\Delta BEP \cong \Delta DEQ$ (AAS),
$\therefore$ PE = QE (corr. sides, $\cong\Delta$s)
16. In ΔAPD and ΔCPD, PD = PD (common), AD = CD (prop. of rhombus),
$\angle ADP = \angle CDP$ (prop. of rhombus), $\therefore$ $\Delta APD \cong \Delta CPD$ (SAS),
$\therefore$ $\angle APD = \angle CPD$ (corr. $\angle$s, $\cong\Delta$s), i.e. PB bisects $\angle APC$.
17. AE = $y - 1$ (rhombus), $(y+1)^2 + (y-1)^2 = 10^2$ (Pyth. Thm.)
$y^2 + 2y + 1 + y^2 - 2y + 1 = 100$, $2y^2 = 98$, $y^2 = 49$, $\therefore$ $y = \sqrt{49} = 7$
$x = y + 1 = 7 + 1 = 8$ (rhombus)
18. (a) $\angle A = \angle BEA$ (base $\angle$s, isos. Δ), $\angle A = \dfrac{180° - 36°}{2} = 72°$ ($\angle$ sum of Δ),
$\angle C = \angle A = 72°$ (prop. of //gram); $\angle BDC = \angle C = 72°$ (base $\angle$s, isos. Δ),
$\theta + 36° = \angle BDC = 72°$ (alt. $\angle$s, AB // DC), $\therefore$ $\theta = 36°$
(b) $\angle EDB = 180° - \angle A - \theta - 36° = 180° - 72° - 36° - 36° = 36°$ ($\angle$ sum of Δ),
$\because$ $\angle EDB = \theta = 36°$, $\therefore$ BE = DE (sides opp. eq. $\angle$s),
$\therefore$ ΔBED is isosceles.
19. (a) $\angle SQN = 90° + 45° = 135°$ (prop. of square),
$\angle NQM = \angle SQM = 135° \div 2 = 67.5°$, $\angle QPM = 45°$ (prop. of square),
$\therefore$ $\angle M = 67.5° - 45° = 22.5°$ (ext. $\angle$ of Δ)
(b) $\angle SQR = 45°$ (prop. of square), $\angle RQM = 67.5° - 45° = 22.5° = \angle M$,
$\therefore$ RM = RQ (sides opp. eq. $\angle$ s), $\therefore$ ΔQRM is isosceles.
20. AC = BC and $\angle ACB = 60°$ (equilateral Δ), $\therefore$ $\angle ACD = 60° + 90° = 150°$,
$\therefore \angle CDA = \angle CAD$ (base $\angle$s, isos. Δ),
$\angle CDA = \dfrac{180° - 150°}{2} = 15°$ ($\angle$ sum of Δ), but $\angle CDB = 45°$ (prop. of square),
$\therefore$ $a = 45° - 15° = 30°$; $\angle ADE = 90° - 15° = 75°$
Similarly, $\angle AED = 75°$, $\therefore$ $b + 75° + 75° = 180°$ ($\angle$ sum of Δ), $\therefore$ $b = 30°$
21. (a) In ΔPQT and ΔRQT, PQ = RQ (prop. of rhombus),
$\angle PQT = \angle RQT$ (prop. of rhombus), QT = QT (common),
$\therefore$ $\Delta PQT \cong \Delta RQT$ (S.A.S.)
(b) $\because$ $\Delta PQT \cong \Delta RQT$ (proved), $\therefore$ $\angle QTP = \angle QTR$ (corr. $\angle$s, $\cong\Delta$s),
$\angle QTP = 70° \div 2 = 35°$, $\angle PQT = 180° - 35° - 90° = 55°$ ($\angle$ sum of Δ),
but $\angle RQS = \angle RSQ = \angle PQT = 55°$ (prop. of rhombus),
$\therefore$ $\angle QRS = 180° - 55° - 55° = 70°$ ($\angle$ sum of Δ)
22. In ΔAED and ΔCGD, AD = CD and DE = DG (prop. of square),

$\angle ADC = \angle EDG = 90°$ (prop. of square),
$\angle ADE = \angle CDE + \angle ADC = \angle CDE + 90° = \angle CDE + \angle EDG = \angle CDG$,
$\therefore$ $\Delta AED \cong \Delta CGD$ (S.A.S.), $\therefore$ AE = CG (corr. sides, $\cong\Delta$s)

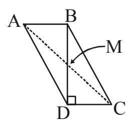
23. QN = NS (prop. of //gram), NS = LP (prop. of //gram), $\therefore$ QN = LP;
LP // NS (def. of //gram), i.e. LP // QS. $\therefore$ QNPL is a //gram (2 sides eq. and //),
$\therefore$ LN and QP bisect each other (prop. of //gram), $\therefore$ LM = MN

24. (a) In //gram PQRS, let QS and PR intersect at X,
PX = XR and QX = XS (prop. of //gram).
In //gram QTSU, let QS and TU intersect at Y,
QY = YS and TY = YU (prop. of //gram).
Since QX = XS and QY = YS, X and Y are the same point,
i.e. QS, PR and TU are concurrent.
(b) In quadrilateral PTRU, X is the mid-point of TU and PR (proved),
$\therefore$ PTRU is a //gram (diags bisect each other)

25. $\angle CBF = 28°$ (prop. of rectangle), $\angle BFC = 180 - 28° - 28° = 124°$ ($\angle$ sum of Δ),
$\angle AFE = 60°$ (equilateral Δ), $\angle DFE + 60° = 124°$ (vert. opp. $\angle$s), $\angle DFE = 64°$,
$\because$ DF = AF (prop. of rectangle) and AF = EF (equilateral Δ), $\therefore$ DF = EF,
$\therefore$ $\angle FED = \angle FDE$ (base $\angle$s, isos. Δ), $\therefore$ In ΔEFD, $\angle FED = \dfrac{180° - 64°}{2} = 58°$,
$\therefore$ $\angle AED = \angle AEF + 58° = 60° + 58° = 118°$

26. (a) $\angle CDE = (5-2)\times 180° \times \dfrac{1}{5} = 108°$ ($\angle$sum of polygon)
In ΔCDE, $\angle DCE = \angle DEC$ (base $\angle$s, isos. Δ),
$\therefore$ $\angle DCE = \dfrac{180° - 108°}{2} = 36°$ ($\angle$ sum of isos. Δ)
(b) $\angle BCD = \angle CDE = 108°$ (prop. of regular pentagon),
but $\angle BCF = \angle DCF$ (prop. of rhombus),
$\therefore$ $\angle DCF = 108° \div 2 = 54°$, $\therefore$ $\theta = 54° - 36° = 18°$

27. DC = AB = 7 (prop. of // gram),



$\therefore$ BD = $\sqrt{25^2 - 7^2} = \sqrt{576} = 24$ (Pyth. Thm.),
DM = BM = $24 \div 2 = 12$ (prop. of //gram),
$\therefore$ CM = $\sqrt{7^2 + 12^2} = \sqrt{193}$ (Pyth. Thm.),
$\because$ AM = AC = $\sqrt{193}$ (prop. of //gram), $\therefore$ AC = $2\sqrt{193}$ = 27.8
[*Method 2*: Produce CD to E such that AE $\perp$ CE. ED = AB = 7, AE = BD = 24,
$\therefore$ AC = $\sqrt{(7+7)^2 + 24^2} = \sqrt{772}$ = 27.8]

28. (a) In ΔAXM and ΔCYM, $\angle MAX = \angle MCY$ (alt.$\angle$s, AB // DC),
$\angle AMX = \angle CMY = 90°$ (vert. opp. $\angle$s), MX = MY (given),
$\therefore$ $\Delta AXM \cong \Delta CYM$ (AAS)
(b) In ΔXM and ΔCAD, $\angle MAX = \angle DCA$ (alt. $\angle$s, AB // DC),
$\angle AMX = \angle CDA = 90°$, $\angle AXM = \angle CAD$ ($\angle$ sum of Δ),
$\therefore$ $\Delta AXM \sim \Delta CAD$ (AAA)

(c) $AC = \sqrt{5^2+12^2} = \sqrt{169} = 13$ cm (Pyth. Thm.), $\because$ $\Delta AXM \cong \Delta CYM$ (proved),
$\therefore$ AM = CM (corr. sides, $\cong\Delta$s), $\therefore$ $AM = 13 \div 2 = 6.5$ cm,
$\because$ $\Delta AXM \sim \Delta CAD$ (proved), $\therefore$ $\frac{XM}{AD} = \frac{AM}{CD}$, $XM = \frac{6.5}{12} \times 5 = \frac{65}{24}$,
$\therefore$ Area of $\Delta AXM = \frac{1}{2}(6.5)(\frac{65}{24}) = 8.8$ cm^2

29. (a) $\because$ AB = BD (given), $\therefore$ $\angle ADB = \angle DAB$ (base $\angle$s, isos. Δ),
$\angle ABD = \angle BDC$ (alt. $\angle$s, AB // DC),
$\therefore$ $\angle CDA = \angle BDC + \angle ADB = \angle ABD + \angle DAB = \angle EDA$ (ext. $\angle$ of Δ),
(b) In ΔACD and ΔAED, AD = AD (common), DC = AB (prop. of //gram),
$\therefore$ DC = AB = BD = DE, CDA = $\angle EDA$ (proved),
$\therefore$ $\Delta ACD \cong \Delta AED$ (SAS), $\therefore$ AC = AE (corr. sides $\cong\Delta$s),
$\therefore$ ΔACE is isosceles.

30. (a) In ΔACE and ΔABD, AC = AB and AE = AD (equilateral Δs),
$\angle EAC = 60° + \angle DAC = \angle DAB$, $\therefore$ $\angle EAC = \angle DAB$,
$\therefore$ $\Delta ACE \cong \Delta ABD$ (SAS), $\therefore$ EC = BD (corr. sides, $\cong\Delta$s)
(b) In ΔACE and ΔDFE, AE = DE and CE = FE (equilateral Δs),
$\angle AEC = 60° - \angle CED = \angle DEF$, $\therefore$ $\angle AEC = \angle DEF$,
$\therefore$ $\Delta ACE \cong \Delta DEF$ (SAS), $\therefore$ AC = DF (corr. sides, $\cong\Delta$s)
(c) $\because$ BD = EC (proved) and EC = CF (equilateral Δ), $\therefore$ BD = CF,
$\because$ AC = DF (proved) and AC = BC (equilateral Δ), $\therefore$ DF = BC,
$\therefore$ BDFC is a //gram (opp. sides eq.)

31. (a) $\angle BPS = \angle BSP = \angle BRS = 45°$ (prop. of square),
$\angle BPA = \angle SPA = 45° \div 2 = 22.5°$, $\angle ABP = \angle BRS = 45°$ (corr. $\angle$s, AB // SR),
$\therefore \angle BAC = \angle BPA + \angle ABP = 22.5° + 45° = 67.5°$ (ext. $\angle$ of Δ) and,
$\angle BCA = \angle BSP + \angle SPA = 45° + 22.5° = 67.5°$ (ext.$\angle$ of Δ),
$\therefore$ $\angle BAC = \angle BCA$, $\therefore$ BA = BC (sides opp., eq. $\angle$s)
(b) In ΔPBA and ΔPRT, $\angle BPA = \angle RPT$ (common),
$\angle PBA = \angle PRT$ and $\angle PAB = \angle PTR$ (corr. $\angle$s, AB // SR),
$\therefore$ $\Delta PBA \sim \Delta PRT$ (AAA)
(c) $\because$ $PB = \frac{1}{2}PR$ (prop. of square), $\therefore$ $\frac{PB}{PR} = \frac{1}{2}$,
$\because$ $\Delta PBA \sim \Delta PRT$ (proved), $\therefore$ $\frac{BA}{RT} = \frac{PB}{PR} = \frac{1}{2}$ (corr. sides, ~ Δs),
but BA = BC (proved), $\therefore$ $\frac{BC}{RT} = \frac{1}{2}$, $\therefore$ 2BC = RT.

32. In ΔGBF and ΔCD, $\angle G = \angle G$ (common), $\angle GBF = \angle GCD$ and
$\angle GFB = \angle GDC$ (corr. $\angle$s, BF // CD), $\therefore$ $\Delta GBF \sim \Delta GCD$ (AAA),
$\therefore$ $\frac{BF}{CD} = \frac{GF}{GD} = \frac{2}{2+1} = \frac{2}{3}$ (corr. sides, ~Δs), 3BF = 2CD, but BA = CD (prop. of //gram),

3(BA – FA) = 2CD, 3CD – 3FA = 2CD, $\therefore$ CD = 3FA, $\frac{CD}{FA} = 3$.

In ΔCDE and ΔAFE, ∠DEC = ∠FEA (vert. opp. ∠s), ∠CDE = ∠AFE and ∠DCE = ∠FAE (alt. ∠s, BA // CD), $\therefore$ ΔCDE ~ ΔAFE (AAA),

$\therefore \frac{DE}{EF} = \frac{CD}{FA} = 3$ (corr. sides, ~Δs), $\therefore$ DE : EF = 3 : 1

33. $\frac{RT}{RP} = \frac{RV}{RS}$ (given), ∠TRV = ∠PRS (common ∠),

$\therefore$ ΔTRV ~ ΔPRS (AAA), $\therefore$ ∠VTR = ∠SPR (corr. ∠s, ~Δs)

$\therefore$ TV // PS. (corr. ∠s equal)

But PS // QR (def. of //gram), i.e. PS // UR, $\therefore$ TV // UR .....(1)

$\frac{RT}{RP} = \frac{RU}{RQ}$ (given), ∠TRU = ∠PRQ (common ∠),

$\therefore$ ΔTRU ~ ΔPRQ (AAA), $\therefore$ ∠RUT = ∠RQP (corr. ∠s, ~Δs)

$\therefore$ TV // UR (corr. ∠s equal)

But PQ // SR def. of //gram), i.e. PQ // VR, $\therefore$ TU // VR .....(2)

$\therefore$ From (1) and (2), RUTV is a parallelogram. (by definition)

34. (a) ∠PQS = 45° (prop. of square),

∠UPQ + ∠PUQ = ∠PQS (ext. ∠ of Δ),

∠UPQ Z + 22° = 45°, ∠UPQ = 23° ≠ ∠PVQ, $\therefore$ PQ ≠ UQ

The claim is disagreed.

(b) (i) PS = PQ = 4 cm and ∠QPS = 90° (prop. of square)

$\therefore$ QS = $\sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$ (pyth. thm.)

QT = $\frac{1}{2}QS = \frac{1}{2}\left(4\sqrt{2}\right) = 2\sqrt{2}$ cm (prop. of square)

(ii) PQ ⊥ QS (prop. of square), $\therefore$ ∠PTQ = 90°, i.e. ∠PTU = 90°,

PT = QT = $2\sqrt{2}$ cm (diagonals equal and bisect each other)

In ΔPTU, $\tan 22° = \frac{PT}{TU}$, $\therefore$ TU = $\frac{2\sqrt{2}}{\tan 22°}$

QU = TU – QT = $\frac{2\sqrt{2}}{\tan 22°} - 2\sqrt{2}$ = 4.17 cm

35. (a) DE = DG (given), DA = DC (def. of square),

∠DAE = ∠DCF = 90° (def. of square),

∠DCG + ∠DCF = 180° (adj. ∠s on a st. line),

∠DCG = 180° – 90° = 90°, $\therefore$ ∠DCG = ∠DAE,

$\therefore$ ΔADE ≅ ΔCDG (RHS)

(b) ∠CDA = 90° (prop. of square),

∠ADE = ∠CDG (corr. ∠s, ΔADE ≅ ΔCDG).

Let x = ∠ADE = ∠CDG.

∠GDP = ∠CDG + (∠CDA – 45° – ∠ADE) = x + (90° – 45° – x) = 45° = ∠EDF.

DE = DG (given), DF = DF (common side)

$\therefore$ ΔDEF $\cong$ ΔDGF (SAS)

(c) $\angle$BGD = 66° (given), $\therefore$ $\angle$DGF = 66°,

$\angle$DEF = $\angle$DGF = 66° (corr. $\angle$s, ΔDEF $\cong$ ΔDGF).

In ΔDEF, $\angle$DFE = 180° − 45° − 66° = 69° ($\angle$ sum of Δ).

$\angle$DFG = $\angle$DFE = 69° (corr. $\angle$s, ΔDEF $\cong$ ΔDGF)

$\angle$BFE = 180° − $\angle$DFG − $\angle$DFE (adj. $\angle$s on a st. line)

= 180° − 69° − 69° = 42°

36. (a) TQ $\perp$ PS (rhombus), $\therefore$ $\angle$TVS = 90°.

PS // QR (//gram), $\therefore$ $\angle$TQR = $\angle$TVS = 90° (corr. $\angle$s, PS // QR).

$\therefore$ ΔQTR is a right-angled triangle.

(b) TV = VQ (rhombus), area of ΔTVS = area of ΔQVS,

$\therefore$ area of ΔQST = 2 × (area of ΔTVS), note that ΔSVT ~ ΔRQT (AAA).

$\therefore$ area of ΔVST : area of ΔQRT = $VT^2 : QT^2 = 1^2 : 2^2$ = 1 : 4

$\therefore$ area of ΔSVT : area of QRSV = 1: (4 − 1) = 1 : 3

$\therefore$ area of ΔQST : area of QRSV = 1 × 2 : 3 = 2 : 3

(c) QS = 2QU = 2 (10) = 20 cm (//gram)

TS = QS = 20 cm (rhombus)

SQ = PQ (//gram), and PC = TS (rhombus)

$\therefore$ AR = TS = 20 cm

TR = TS + SR = 20 + 20 = 40 cm

$TQ = \sqrt{TS^2 - QR^2} = \sqrt{40^2 - 24^2} = 32$ (Pyth. thm.)

Area of $\Delta QTR = \frac{1}{2}(TQ)(QR) = \frac{1}{2}(32)(24) = 384 cm^2$

Area of ΔVST : area of ΔQTR = 1 : 4 (proved)

$\therefore$ area of $\Delta VST = \frac{1}{4}(384) = 96\ cm^2$

area of ΔQTS = 2 × (area of ΔTVS) = 2 (96) = 192 cm^2

Unit 7 Mid-point theorem & intercept theorem

1. (a) $\because$ AB // CD // EF and BD = DF, $\therefore$ $x = 5$ (intercept thm), $y = 6$ (intercept thm)

(b) $\because$ AC = CE and AD = DF (given), $\therefore$ $x = \frac{1}{2} \times 20 = 10$ (mid-pt thm)

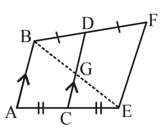
$\because$ AC = CE and BD = DE (given), $\therefore$ $y = 2(10) = 20$ (mid-pt thm)

2. (a) $\because$ AC = CE and AB // CD (given),

$\therefore$ BG = GE (intercept thm),

$\therefore$ $CG = \frac{1}{2} \times 4 = 2$ (mid-pt thm),

$DG = 7 - 2 = 5$, $a = 2(5) = 10$ (mid-pt thm)

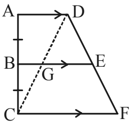


(b) ∵ AB = BC and AD // BE (given),

∴ DG = GC and DE = EF (intercept thm),

∴ BG = $\frac{1}{2}\times 10 = 5$ (mid-pt thm), GE = $\frac{1}{2}\times 18 = 9$ (mid-pt thm),

∴ $y = 5 + 9 = 14$



3. (a) AC = AB = 16 cm (sides opp. eq. ∠s). ∵ DG = GA and DF = FC (given),

∴ EF//AC and GF = $\frac{1}{2}(16) = 8$ (mid-pt thm).

∵ BE = EA (given) and EF // AC (proved), ∴ BG = GC (intercept thm)

∴ $r = \frac{1}{2}\times 16 = 8$ (mid-pt thm)

(b) ∵ AE = EF and AB // EG (given), ∴ BD = DF (intercept thm),

∴ AB = 2(3) = 6 (mid-pt thm). ∠BAD = ∠GDC (corr. ∠s, AB // EG),

∠GDC = ∠GCD (given), ∴ ∠BAD = ∠GCD,

∴ y = AB = 6 (sides opp. eq. ∠s)

4. (a) ∵ AD = DC and BE = EC (given), ∴ AB // DE and $y = 2x$ (mid-pt thm),

∴ $a°$ = ∠B = 90° (corr. ∠s, AB // DE), ∴ $a = 90$

∠CDE = 180° − 90° − 45° = 45° (∠ sum of Δ)

∵ ∠CDE = ∠C = 45°, ∴ x = CE = 6 (sides opp. eq. ∠s). $y = 2(6) = 12$

(b) PQ = 2(3) = 6, QR = $\sqrt{10^2 - 6^2} = \sqrt{64} = 8$ (Pyth. thm.),

∵ QS = SP and ST//PR (given), ∴ y = QT = $\frac{1}{2}\times 8 = 4$ (intercept thm)

$x = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ (Pyth. thm.)

5. EF // CD // AB and EC = CA, ∴ AG = GF and BD = DE (intercept thm),

∴ CG = $\frac{1}{2}\times 9 = 4.5$ cm (mid-pt thm). CD = $\frac{1}{2}\times 15 = 7.5$ (mid-pt thm),

∴ GD = $7.5 - 4.5 = 3$ cm

6. ∵ AB = BD and AC = CE (given), ∴ BC // DE (mid-pt thm),

∴ BCED is a trapezium.

7. ∵ QH = HR and RN // HK (given), ∴ NK = KQ (intercept thm.)

∵ PM = MH and RN // HK (given), ∴ PN = NK (intercept thm.), ∴ PN = KQ

8. ∵ AB = BC and BF // CE (given), ∴ AF = FE (intercept thm)

∵ AB = BC and BE // CD (given), ∴ AE = ED (intercept thm)

Let AF = x, ∴ FE = x, ED = $x + x = 2x$, AF : FE : ED = $x : x : 2x$ = 1 : 1 : 2

9. Join PR. ∵ PA = AQ and RB = BQ (given),

∴ AB = $\frac{1}{2}$PR and PR // AB (mid-pt thm).

∵ PD = DS and RC = CS (given), ∴ DC = $\frac{1}{2}$PR and PR // DC (mid-pt thm),

AB = DC and AB // DC (proved), ∴ ABCD is a //gram (2 sides eq. and //)

10. Join DE. ∵ BE = EC and AD = DC (given),

$\therefore$ AB // DE and AB = 2ED (mid-pt thm), $\dfrac{AB}{ED} = \dfrac{2}{1}$.

$\because$ $\Delta ABG \sim \Delta EDG$ (AAA), $\therefore$ $\dfrac{BG}{GD} = \dfrac{AG}{GE} = \dfrac{AB}{ED} = \dfrac{2}{1}$ (corr. sides, ~Δs),

$\therefore$ BG : GD = 2 : 1, and AG : GE = 2 : 1

11. (a) $\because$ PA = AQ and PB = BR (given), $\therefore$ AD // QR (mid-pt thm)

 (b) $\because$ PA = AQ (given) and AD // QC (proved), $\therefore$ PD = DC (intercept thm),

 $\therefore$ AD = $\dfrac{1}{2}$QC (mid-pt thm).

 $\because$ PB = BR (given) and PD = DC (proved), $\therefore$ DB = $\dfrac{1}{2}$CR

 But QC = CR, $\therefore$ AD : DB = $\dfrac{1}{2}$QR : $\dfrac{1}{2}$CR = 1 : 1

12. HP // KQ and MP = PQ, $\therefore$ MH = HK (intercept thm.). KR // HP and NR = RP,

 $\therefore$ NK = HK (intercept thm.). t cm = NK = HK = MH = 5 cm, $t = 5$.

 KR = $\dfrac{1}{2}$HP = 3 cm (mid-pt. thm.). KQ = 2HP = 12cm (mid-pt. thm.)

 $\therefore$ $3 + x = 12$, $x = 9$

13. $\because$ ∠SQT = ∠QPR (given), $\because$ QT // PU (corr. ∠s eq.), QR // TU (given),

 $\therefore$ RQTU is a //gram, $\therefore$ RU = 7 cm (prop. of //gram).

 $\because$ PQ = QS and QR//SU (given), $\therefore$ $a = 7$ (intercept thm)

 ∠PQR = ∠S (corr.∠ s, QR//SU), but ∠S = ∠P, $\therefore$ ∠PQR = ∠P,

 $\therefore$ QR = 7 (sides opp. eq. ∠s), $\therefore$ $b = 7$ (prop. of //gram)

14. (a) $\because$ QL = LP = 10 and QM = MR = 15, $\therefore$ LM // PR (mid-pt. thm.)

 (b) $z = 2(14) = 28$ (mid-pt. thm.). $\because$ ΔLMN ~ ΔRPN (AAA),

 $\therefore$ $\dfrac{x}{6} = \dfrac{y}{13} = \dfrac{28}{14} = 2$ (corr. sides, ~Δs), $\therefore$ $x = 2 \times 6 = 12$ and $y = 2 \times 13 = 26$

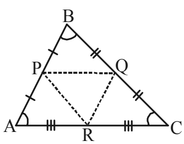
15. $\because$ BP = PA and BQ = QC (given), $\therefore$ PQ = $\dfrac{1}{2} \times$ AC and PQ // AC (mid-pt. thm.)

 $\because$ DS = SA and DR = RC (given), $\therefore$ SR = $\dfrac{1}{2} \times$ AC and SR // AC (mid-pt. thm.)

 $\therefore$ PQ = SR and PQ // SR, $\therefore$ PQRS is a //gram (2 sides eq. and //)

 $\therefore$ PS = QR (prop. of //gram)

16. Let P, Q, R be the mid-points of AB, BC and AC respectively.
 Join PQ, QR and PR.



 $\because$ BP = PA and BQ = QC (construction),

 $\therefore$ PQ = $\dfrac{1}{2}$AC (mid-pt thm), $\therefore$ PQ = AR = RC.

 Similarly, QR = BP = PA and PR = BQ = QC.

 $\therefore$ ΔBPQ ≅ ΔPAR ≅ ΔQRC ≅ ΔRQP (SSS)

17. (a) $\because$ RA = AQ and RB = BP (given), $\therefore$ AB = $\dfrac{1}{2}$PQ (mid-pt. thm.),

$\because$ PC = CS and PB = BR (given), $\therefore$ BC = $\frac{1}{2}$RS (mid-pt. thm.)

But PQ = RS (given), $\therefore$ AB = BC, $\therefore$ ΔABC is isosceles

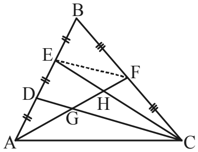
(b) $\angle$QPR = 1180° – 95° – 15° = 70° ($\angle$ sum of Δ)

$\because$ AB // QP (mid-pt. thm.), $\therefore$ $\angle$ABR = 70° (corr. $\angle$s, AB // QP).

BC // RS (mid-pt. thm.), $\therefore$ $\angle$RBC + 100° = 180° (int. $\angle$s, BC // RS), $\angle$RBC = 80°

$\therefore$ $\angle$ ABC = 70° + 80° = 150°

18. Join EF. BE = ED and BF = FC,
$\therefore$ DC = 2EF and DC//EF (mid-pt. thm.)
AD = DE and DC//EF (proved),
$\therefore$ AG = GF (intercept thm.), $\therefore$ EF = 2 DG (mid-pt. thm.)
Let DG = x, $\therefore$ EF = $2x$, DC = $4x$, GC = $4x - x = 3x$.
$\therefore$ DG : GC = $x : 3x$ = 1 : 3



19. (a) In ΔKHP and ΔKHQ, KH = KH (common), $\angle$KHP = $\angle$KHQ = 90° (given),
$\angle$HKP = $\angle$HKQ (given), $\therefore$ ΔKHP $\cong$ ΔKHQ (ASA),
$\therefore$ HP = HQ (corr. sides, $\cong$ Δs), SP = SR (given),
$\therefore$ HS // QR (mid-pt. thm.)

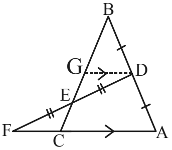
(b) $\because$ HP = HQ and HS // QR (proved), $\therefore$ PL = LK (intercept thm.),
$\therefore$ HL = $\frac{1}{2}$QK (mid-pt. thm.), but QK = PK (corr. sides, $\cong$ Δs), $\therefore$ HL = $\frac{1}{2}$PK

20. (a) $\because$ QH = HR and PK = KR (given), $\therefore$ HK // QP (mid-pt. thm.),
$\therefore$ $\angle$KHR = 90° (corr. $\angle$s, HK // QP),
$\therefore$ $\angle$KHQ = 180° – 90° = 90° (adj. $\angle$s on st. line).
In ΔQHK and RHK, QH = RH (given), KH = KH (common),
$\therefore$ $\angle$KHQ = $\angle$KHR = 90°, $\therefore$ ΔQHK $\cong$ ΔRHK (SAS),
$\therefore$ QK = RK (corr. sides, $\cong$ Δs), $\therefore$ ΔRKQ is isosceles

(b) $\because$ QK = RK (proved) and PK = PK (given), $\therefore$ PK = QK,
$\therefore$ ΔPKQ is isosceles

21. In ΔDEB and ΔDEF, BD = DF = 7cm (given), DE = DE (common),
$\angle$DEB = $\angle$DEF = 90° (given), $\therefore$ ΔDEB $\cong$ ΔDEF (RHS)
$\therefore$ EF = BE = 5cm (corr. sides, $\cong$ Δs). $\because$ $\angle$DEF = $\angle$AFC = 90° (given),
$\therefore$ DE//AF (corr. $\angle$s eq.), BE = EF = 5cm (proved),
$\therefore$ DA = BD = 7cm (intercept thm.).
$\because$ BD = DA (proved) and DF // AC (given),
$\therefore$ FC = BF = 5 + 5 = 10 cm (intercept thm.)

22. (a) Draw GD // FA,
$\because$ BD = DA (given) and GD // FA (construction),
$\therefore$ BG = GC (intercept thm.), $\therefore$ AC = 2DG (mid-pt. thm.),
In ΔDEG and ΔFEC, DE = FE (given),
$\angle$DEG = $\angle$FEC (vert. opp. $\angle$s),
$\angle$ GDE = $\angle$CFE (alt. $\angle$s, GD // FC), $\therefore$ ΔDEG $\cong$ ΔFEC (ASA),



$\therefore$ DG = FC (corr. sides, $\cong \Delta$s), $\frac{AC}{FC} = \frac{2DG}{DG} = 2$, $\therefore$ AC : FC = 2 : 1

(b) $\because$ $\Delta DEG \cong \Delta FCE$ (proved), $\therefore$ GE = CE (corr. sides, $\cong \Delta$s),
BE = BG + GE = GC + GE = EC + GE + GE = 3EC, $\therefore$ BE : EC = 3: 1

23. (a) In ΔBAD and ΔBCD, BD = BD (common), $\angle BDA = \angle BDC = 90°$ (given),
$\angle ABD = \angle CBD$ (given), $\therefore$ $\Delta BAD \cong \Delta BCD$ (ASA),
$\therefore$ AD = CD (corr. sides, $\cong \Delta$s), i.e. D is the mid-pt. of AC

(b) $\because$ AE = EF (given) and AD = DC (proved), $\therefore$ DE // CF (mid-pt thm)

(c) DE = $\frac{1}{2}$CF (mid-pt thm), i.e. DE = $\frac{1}{2}$(BC – BF),

but AB = BC (corr. sides, $\cong \Delta$s), $\therefore$ DE = $\frac{1}{2}$(AB – BF)

24. (a) $\because$ AM = MB and CN = NB (given), $\therefore$ MN // AC (mid-pt. thm.)
In ΔABC and ΔMBN, $\angle B = \angle B$ (common),
$\angle BAC = \angle BMN$ (corr. $\angle$s, MN // AC),
$\angle BCA = \angle BNM$ (corr. $\angle$s, MN //AC), $\therefore$ $\Delta ABC \sim \Delta MBN$ (AAA)
In ΔABC and ΔCBM, $\angle B = \angle B$ (common), $\angle BAC = \angle BCM$ (given),
$\angle BCA$ and $\angle BMC$ ($\angle$ of Δ), $\therefore$ $\Delta ABC \sim \Delta CBM$ (AAA)
Ans. ΔMBN and ΔCBM are similar to ΔABC.

(b) $\because$ $\Delta ABC \sim \Delta CBM$ (proved), $\therefore$ $\frac{BA}{BC} = \frac{BC}{BM}$ (corr. sides, ~ Δs),
$\therefore$ $BC^2 = BM \times BA$

(c) BA = 2BM and BC = 2BN (given), $\because$ $BC^2 = BM \times BA$,
$\therefore$ $(2BN)^2 = BM \times 2BM$, $4BN^2 = 2BM^2$, $2 = \frac{BM^2}{BN^2}$, $\frac{BM}{BN} = \sqrt{2}$,
$\therefore$ BM : BN = $\sqrt{2}$: 1

25. (a) $\angle PTQ = 90°$ (prop. of square)
$\angle QTW = \frac{1}{2}\angle PTQ = \frac{1}{2}(90°) = 45°$ (given)
$\angle PSQ = 45° = \angle QTW$ (prop. of square)
$\therefore$ TW // PS (corr. $\angle$s equal), PS // QR (def. of //gram) , $\therefore$ TQ // QR.

(b) QT = TS (prop. of square), $\therefore$ QW = WU (intercept thm.) ,
$\therefore$ TW = $\frac{1}{2}$SU (mid-pt. thm.)

(c) UV = VS (given), TW = $\frac{1}{2}$SU = $\frac{1}{2}$(UV + VS) = $\frac{1}{2}$(2UV) = UV ,
$\therefore$ UVTW is a parallelogram. (2 sides equal and //)
$\therefore$ TV = WU (prop. of //gram), TV = QW (proved), TV = 8 cm

26. (a) $\because$ BC = ED (opp. sides, //gram)
BF = $\frac{1}{2}$BC and EG = $\frac{1}{2}$ED (given)

∴ BF = EG, BC // ED (def. of //gram), ∴ BF // EG ,

∴ BEGF is a parallelogram (opp. sides equal and //)

(b) (i) ED = 2EG (given), AE : EG = 2 : 1 (given), ∴ AE = 2EG

∴ ED = AE, AE : ED = 1: 1

(ii) ∵ BC = ED (opp. sides of //gram), ED = AE (proved) ,

∴ BC = AE, ∠CBH = ∠AEH (alt. ∠s, BC // AD) ,

∠BHC = ∠EHA (vert. opp. ∠s),

∴ ΔCBH ≅ ΔAEH (AAS), ∴ BH = HE (corr. sides, ≅Δs).

i.e. H is the mid-pt. of BE.

(c) AE = ED (proved), BE // CD (def. of //gram) ,

∴ AH = HC (intercept thm.), ∴ HE = $\frac{1}{2}$CD (mid-pt. thm.), HE : CD = 1 : 2

(d) (i) FG // BE (def. of //gram), BF = FC (given) ,

∴ HI = IC (intercept thm.), i.e. I is the mid-pt. of HC.

(ii) F is the mid-pt. of BC (given), I is the mid-pt. of HC (proved) ,

∴ FI = $\frac{1}{2}$BH (mid-pt. thm.), FI : BH = 1: 2

(e) (i) CD = AB = 4 cm (given), HE : CD = 1 : 2 (proved) ,

∴ HE = $\frac{1}{2}(4) = 2$ cm.

(ii) BE = HE = 2 cm (proved), FI : BH = 1 : 2 (proved) ,

∴ FI = $\frac{1}{2}(2) = 1$ cm.

Unit 8 Centres in a triangle

1. (a) AG; BAC (b) DG; AB
2. (a) PQ = PR, $28 - x = 3x + 8$, $20 = 4x$, $x = 5$

   (b) Let E be a point on CB such that DE ⊥ CB. DE = DA (∠ bisector property),

   ∴ Area of ΔBCD = 21 × 6 ÷ 2 = 63 cm²
3. (a) AP = BP (prop. of ⊥ bisector), $5x + 10 = 7x - 26$, $36 = 2x$, $x = 18$

   (b) AM = BM (converse of ⊥ bisector property), $2y - 11 = 34 - 3y$, $5y = 45$, $y = 9$;

   PB = PA = 9 + 7 = 16 cm
4. (a) orthocentre (b) in-centre (c) circumcentre (d) centroid
5. AC, BC and CD
6. ∠SPQ = 180° − 46° − 113° = 21° (∠ sum of Δ), ∴ ∠SPQ ≠ ∠SPR,

   ∠SQR = 180° − 110° − 24° = 46° (∠ sum of Δ), ∴ ∠SQR = ∠SQP,

   ∠SRP = 180° − 19° − 21° − 46° − 46° − 24° = 24° (∠sum of Δ),

   ∴ ∠SRP = ∠SRQ, ∴ *QS and RS are angle bisectors.*
7. Let ∠ABD = a = ∠CBD. ∠BCA= 180°− 90° − ∠CBD (∠ sum of Δ), = 90° − a

   ∠BAC = 180°− 90° − ∠ABD (∠ sum of Δ), = 90° − a

   ∵ ∠DCA = ∠BAC (alt. ∠s, AB // DC), ∴ ∠DCA = 90° − a

$\because$ ∠DCA = ∠BCA, $\therefore$ *AC is an angle bisector of ΔBCD.*

8. (a) AC = $\sqrt{12^2+16^2} = \sqrt{400} = 20$ (Pyth. thm.), $\therefore$ EC = 20 – 10 = 10

(b) $\because$ AE = EC = 10, $\therefore$ *BE is a median of ΔABC.*

(c) If BE ⊥ AC, then in ΔABE, BE = $\sqrt{12^2-10^2} = \sqrt{44}$.

If BE ⊥ AC, then in ΔBEC, BE = $\sqrt{16^2-10^2} = \sqrt{156}$.

However, $\sqrt{44} \neq \sqrt{156}$, "BE ⊥ AC" must be false. $\therefore$ *BE is not an altitude of ΔABC.*

9. (a) $\because$ EF ⊥ CD and EF bisects CD,

$\therefore$ *EF is the perpendicular bisector of CD.*

(b) In ΔCEF and ΔDEF, CF = CF (given), ∠CFE = ∠DFE = 90° (given),

EF = EF (common), $\therefore$ ΔCEF ≅ ΔDEF (SAS), $\therefore$ CE = CE (corr. sides, ≅Δs),

$\therefore$ *ΔCDE is an isosceles triangle.*

10. (a) In ΔPTS and ΔRTS, PS = RS and ∠PST = ∠RST (given), ST = ST (common),

$\therefore$ ΔPTS ≅ ΔRTS (S.A.S.)

(b) $\because$ ΔPTS ≅ ΔRTS (proved), $\therefore$ PT = RT (corr. sides, ≅Δs),

∠PTS = ∠RTS (corr. ∠s, ≅Δs), ∠PTS + ∠RTS = 2∠PTS = 180° (adj. ∠s on st. line),

$\therefore$ ∠PTS = 90°, *i.e. QS is the perpendicular bisector of PR.*

11. $\because$ SM and SR are angle bisectors, $\therefore$ S is the incentre of ΔMNR.

S is equidistant from the 3 side of ΔMNR. $\therefore$ Yes, the claim is agreed.

12. (a) $\because$ AC = BC (given) and CN ⊥ AB (given),

$\therefore$ AN = BN (converse of property of ⊥ bisector)

(b) Let P be a point on AC such that BP ⊥ AC.

AN = $\sqrt{AC^2 - CN^2} = \sqrt{26^2 - 24^2}$ = 10 cm, AB = 2BN = 10 cm

Area of ΔABC = $\dfrac{(AB)(CN)}{2} = \dfrac{(AC)(BP)}{2}$, $\therefore$ BP = $\dfrac{(AB)(CN)}{AC} = \dfrac{(20)(24)}{26} = \dfrac{240}{13}$ cm

13. In ΔACQ and ΔBCQ, AQ = BQ (radii), AC = BC (radii), CQ = CQ (common),

$\therefore$ ΔACQ ≅ ΔBCQ (S.S.S.), $\therefore$ ∠AQC = ∠BQC (corr. ∠s, ≅Δs),

$\therefore$ *QC is the angle bisector of ∠PQR.*

14. (a) $b_1 + a_1 + b_2 + a_2 = 180°$ (∠ sum of Δ), $2a_1 + 2b_1 = 180°$ ($\because$ $a_1 = a_2$, $b_1 = b_2$),

$a_1 + b_1 = 90°$, $\because$ ∠BDC = $a_1 + b_1$ (ext. ∠ or Δ), $\therefore$ ∠BDC = 90°

i.e. BD is an altitude of ΔABC.

(b) ∠ABC = $a_1 + b_2 = a_1 + b_1 = 90°$, $\therefore$ ΔABD is a right-angle triangle.

(c) (i) BD, AB and BC (ii) B

15. (a) $\because$ AE = EB = 5cm, $\therefore$ *EC is a median of ΔABC.*

(b) $\because$ ED ⊥ AC, $\therefore$ *ED is an altitude of ΔAEC, ΔAED and ΔCED.*

(c) $AB^2 = 5^2 = 25$, $EF^2 + BF^2 = 4^2 + 3^2 = 25$,

$\because$ $AB^2 = EF^2 + BF^2$, $\therefore$ ∠BFE = 90^o (converse of Pyth. thm.)

$\therefore$ *EF is an altitude of ΔEBC.*

(d) ED = $\sqrt{5^2 - 2^2} = 21$. $\because$ ED ≠ EF, $\therefore$ ∠DCE ≠ ∠FCE.

$\therefore$ *EC is not an angle bisector of ∠ACB.*

16. Let ∠ECD = *a*. ∠DBC = 90° – ∠ECD = 90° – *a* (ext. ∠ of Δ)

$\angle EDC = \angle ECD = a$ (base $\angle$s, isos. Δ)

$\because$ $90° + \angle BDE + \angle EDC = 180°$ (adj. $\angle$s on st. line)

$\therefore$ $\angle DBE = 90° - \angle EDC = 90° - a$

$\because$ $\angle DBC = 90° - a = \angle BDE$, $\therefore$ BE = ED (side opp., equal $\angle$s)

But ED = EC (given), $\therefore$ BE = EC, $\therefore$ DE is a median of ΔBDC

17. $\angle QPH = \angle RPH$ and $\angle QRH = \angle PRH$ (incentre),

$\angle QPR + \angle QRP + 50° = 180°$ ($\angle$ sum of Δ),

$\therefore$ $2\angle RPH + 2\angle PRH = 130°$, $\angle RPH + \angle PRH = 65°$,

$\angle PHR = 180° - \angle RPH - \angle PRH = 180° - 65° = 115°$ ($\angle$ sum ofΔ)

18. (a) $\angle CPO = 90°$ (circumcentre of Δ),

$\angle PCQ + 90° + 90° + 100° = 360°$ ($\angle$ sum of polygon), $\angle PCQ = 80°$

(b) (i) $\because$ OP = OQ (given) and $\angle CPO = \angle CQO = 90°$ (proved)

$\therefore$ CO is the angle bisector of $\angle$ACB (converse of $\angle$ bisector property).

$\therefore$ The incentre of ΔABC lies on OC produced.

(ii) $CP = \frac{1}{2}AC = \frac{1}{2}(10) = 5$ cm. Join OC. $OC = \sqrt{OP^2 + CP^2} = \sqrt{42+52} = \sqrt{41}$ cm

$OB = OC = \sqrt{41}$ cm (property of $\perp$ bisector)

19. (a) $\because$ $\angle ADH = \angle BEH = 90°$ (orthocentre of Δ), $\angle AHD = \angle BHE$ (vert. opp. $\angle$s)

and $\angle DAH = \angle EBH$ ($\angle$ sum of Δ), $\therefore$ $\Delta ADH \sim \Delta BEH$ (AAA)

(b) $\frac{BE}{AD} = \frac{EH}{DH}$ (corr. sides, ~Δs), $BE = \frac{6}{2}(4) = 12$ cm,

AH $\sqrt{AD^2 + DH^2} = \sqrt{4^2 + 2^2} = \sqrt{20}$ cm

Area of $\Delta ABH = \frac{(AH)(BE)}{2} = \frac{(\sqrt{20})(12)}{2} = 6\sqrt{20} = 12\sqrt{5}$ cm

20. (a) CR = AR, AP = BP, CQ = BQ (centroid of Δ)

AB = 2BP = 2(34) = 68 cm, BC = 2BQ = 2(30) = 60 cm,

AC = 2AR = 2(16) = 32 cm

$BC^2 + AC^2 = 60^2 + 32^2 = 4624 = 68^2 = AB^2$,

$\therefore$ $\angle ACB = 90°$ (converse of Pyth. thm.), *i.e. ΔABC is a right-angled triangle.*

(b) Area of $\Delta ABC = \frac{(AC)(BC)}{2} = \frac{(32)(60)}{2} = 960$ cm^2

Area of $\Delta ABR = \frac{(AR)(BC)}{2} = \frac{(16)(60)}{2} = 480$ cm^2

BG : GR = 2 : 1, GR : BR = 1 : (1 + 2) = 1 : 3

Let N be a point on BR produced such that AN $\perp$ BN.

$\frac{\text{area of } \Delta AGR}{\text{area of } \Delta ABR} = \frac{\frac{1}{2}(GR)(AN)}{\frac{1}{2}(BR)(AN)} = \frac{GR}{BR} = \frac{1}{3}$, area of $\Delta AGR = \frac{1}{3}(480) = 160$ cm^2

21. (a) The perpendicular bisectors DE and DF interest at D, $\therefore$ D is the circumcentre of ΔABC.

$\therefore$ DA, DB, DC are radii of the circumcentre passing through A, B, C.

$\because$ DA = DB and DB = DC (radii)

$\therefore$ BDA and ADC are isosceles triangles.

(b) BD = AD = CD (proved) and BG = CG (given)

$\therefore$ DG $\perp$ BC (converse of property of $\perp$ bisector),

$\therefore$ *DG is the perpendicular bisector of BC.*

(c) AE = $\frac{1}{2}$AB = $24 \times \frac{1}{2} = 12$ cm, and AD is a radius,

$\therefore$ radius = $\sqrt{AE^2 + DE^2} = \sqrt{12^2 + 5^2} = \sqrt{169}$ = 13 cm (Pyth. thm.)

22. (a) I is the incentre , $\therefore \angle PQI = \angle IQS$ and $\angle PRI = \angle IRS$,

but $\angle IQS = \angle IRS$ (base $\angle$s, isos. Δ) , $\therefore 2\angle IQS = 2\angle IRS$,

$\therefore \angle PQR = \angle PRS$, PQ = PR (sides opp., eq. $\angle$s).

(b) IQ = IR (given) and PQ = PR (proved) ,

$\therefore$ PS is the perpendicular bisector of QR (converse of $\perp$ bisector property),

$\therefore$ PS $\perp$ QR.

23. (a) $\because \Delta ADI \sim \Delta DBC$, $\therefore \angle ADI = \angle BDC$ (corr. $\angle$s, $\sim\Delta$s) ,

$\angle ADI + \angle BDC = 180°$ (adj. $\angle$s on st. line) , $2\angle ADI = 180°$,

$\angle ADI = 90°$, $\therefore \angle ADB = 90°$.

(b) Let $\angle DAI = x$. $\because \Delta ADI \sim \Delta BDC$, $\therefore \angle DBC = \angle DAI = x$ (corr. $\angle$s, $\sim\Delta$s) ,

$\angle ABD = \angle DBC = x$ (in-centre of Δ) , $\angle BAI = \angle DAI = x$ (incentre of Δ) ,

$\angle ABD + \angle BAD + \angle ADB = 180°$ ($\angle$ sum of Δ) ,

$x + (x + x) + 90° = 180°$, $3x = 90°$, $x = 30°$.

$\angle CAB = x + x = 60°$, $\angle ABC = x + x = 60°$,

$\angle C = 180° - \angle CAB - \angle ABC = 180° - 60° - 60° = 60°$ ($\angle$ sum of Δ)

$\therefore \Delta$AB = AC (sides opp., eq. $\angle$s) and AB = CB (sides opp., eq. $\angle$s) ,

$\therefore \Delta$ABC is equilateral.

24. (a) $\angle BAE = 141° - 128° = 13°$ (ext. $\angle$ of Δ)

$\angle DAE = \angle BAE = 13°$ (incentre) , $\therefore \angle BAD = 13° + 13° = 26°$,

$\angle BDA = 180° - 128° - \angle BAD = 52° - 26° = 26° = \angle BAD$.

$\therefore$ BD = BA (sides opp., eq. $\angle$s). $\therefore \Delta$ABD is an isosceles triangle.

(b) $\angle BCA = \angle DAC$ (alt. $\angle$s, // lines)

$= 13°$

$= \angle BAC$

$\therefore$ AB = BC (sides opp., eq. $\angle$s)

(c) $\because$ BD = BA (proved) and AB = BC (proved), $\therefore$ BA = BC = BD.

$\therefore$ B is the circumcentre of ΔACD.

25. (a) QS = SR (given) , QT = RT (given) ,

$\therefore$ ST is the perpendicular bisector of QR. (converse of prop. of $\perp$ bisector)

(b) ST is the perpendicular bisector (proved), $\therefore \angle QST = \angle RST = 90°$

$\because \Delta PQR \sim \Delta SQT$ (given) , $\therefore \angle QPR = \angle QST = 90°$ (corr. $\angle$s, $\sim\Delta$s).

$\because \angle QPR = 90° = \angle RST$ and ST = PT ,

$\therefore$ RT is the angle bisector of $\angle$PRQ (converse of prop. of $\angle$ bisector)

(c) Let $\angle PRT = \theta$. $\angle SRT = \angle PRT = \theta$ ($\angle$ bisector).

$\angle SQT = \angle SRT = \theta$ (base $\angle$s, isos. Δ)

$\angle SQT + \angle PRQ + \angle QPR = 180°$ ($\angle$sum of Δ) ,

$\theta + (\theta + \theta) + 90° = 180°$, $3\theta = 90°$, $\theta = 30°$.

In ΔPRT , $\frac{PR}{RT} = \cos \angle PRT$,

$RT = \frac{PR}{\cos\angle PRT} = \frac{9}{\cos 30°} = \frac{9}{\frac{\sqrt{3}}{2}} = \frac{18}{\sqrt{3}} = 6\sqrt{3}$; $QT = RT = 6\sqrt{3}$.

26. (a) AE = CE (prop. of $\perp$ bisector). AE = BE (median of Δ).

$\because$ AE = CE = BE , $\therefore$ E is the circumcentre of ΔABC.

(b) (i) AD = CD ($\perp$ bisector) , AE = BE (median of Δ) ,

$\therefore$ DE // CB (mid-pt. thm). $\therefore$ $\angle ADE = \angle ACB$ (corr. $\angle$s, BE // CB) ,

$\angle ADE = 90°$ ($\perp$ bisector). $\therefore$ $\angle ACB = 90°$.

$\therefore$ ΔABC is a right-angled triangle.

(ii) C.

(c) Let the intersection point of BD and CE be G.

$\because$ AD = DC ($\perp$ bisector) , $\therefore$ BD is a median of ΔABC.

CE is another median (given). $\therefore$ G is the centroid of ΔABC.

G lies on CE, C is the orthocentre, E is the circumcentre.

$\therefore$ The claim is agreed.

27. (a) AB = BC (circumcentre of Δ). $\therefore$ $\angle BAC = \angle ACB$ (base $\angle$s, isos. Δ) ,

$\angle ABC = 90°$ (orthocentre of Δ) , $\angle BAC + \angle ACB + \angle ABC = 180°$ ($\angle$ sum of Δ) ,

$2\angle BAC + 90° = 180°$, $\angle BAC = 45° = \angle ACB$.

$\angle CBD + \angle BEC + \angle ACB = 180°$ ($\angle$ sum of Δ) ,

$\angle CBD + \theta + 45° = 180°$, $\angle CBD = 135° - \theta$.

$\angle ABD = \angle ABC - \angle CBD = 90° - (135° - \theta) = \theta - 45°$.

$\because$ AB = BD (cirumcentre of Δ) , $\therefore$ $\angle ADB = \angle BAD$ (base $\angle$s, isos. Δ).

$\angle ADB + \angle BAD + \angle ABD = 180°$ ($\angle$ sum of Δ) ,

$2\angle ADB + \theta - 45° = 180°$, $\angle ADB = \frac{225° - \theta}{2}$.

(b) If AC $\perp$ BD, then $\theta = \angle BEC = 90°$. $\angle BDA = \frac{225° - 90°}{2} = 67.5°$,

$\angle CBD = 135° - 90° = 45°$, BD = BC (proved) ,

$\therefore$ $\angle BDC = \angle BCD$ (base $\angle$s, isos Δ) ,

$\angle BDC + \angle BCD + \angle CBD = 180°$ ($\angle$ sum of Δ) ,

$2\angle BDC + 45° = 180°$, $\angle BDC = 67.5°$.

$\because$ $\angle BDC = \angle BPA$, $\therefore$ BD is the angle bisector of $\angle$ADC.

$\therefore$ The in-centre of ΔACD lies on BD.

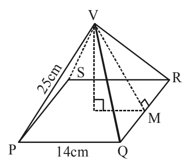
28. (a) $\angle BDC = \angle ADC = 90°$ (altitude of Δ) , BD = AD (centroid of Δ) ,
CD = CD (common side) , $\therefore$ $\Delta BDC \cong \Delta ADC$ (SAS) ,
$\therefore$ BC = AC (corr. sides, $\cong \Delta$s).

(b) (i) Join DE. AD = BD and AE = EC (centroid of Δ) ,
$\therefore$ $DE = \frac{1}{2}BC$ (mid-pt. thm.) and DE // BC (mid-pt. thm.).
$\therefore$ $\angle EDG = \angle BCG$ (alt. $\angle$s, DE // BC) , $\angle DEG = \angle CBG$ (alt. $\angle$s, DE // BC) ,
$\angle DGE = \angle CGB$ (cert. opp. $\angle$s), $\therefore$ $\Delta DGE \sim \Delta CGB$ (AAA).
$\therefore$ $\frac{EG}{GB} = \frac{DE}{BC}$ (corr. sides, ~Δs) , $\frac{EG}{GB} = \frac{\frac{1}{2}BC}{BC} = \frac{1}{2}$, $\therefore$ EG : GB = 1 : 2.

(ii) GB = 2 EG = 2 × 5 = 10. BE = GB + EG = 10 + 5 = 15.

(c) BC = AC = 2 CE = 2 × 12 = 24 , $BE^2 + CE^2 = 15^2 + 12^2 = 369$, $BC^2 = 24^2 = 576$,
$\because$ $BC^2 \neq BE^2 + CE^2$, $\therefore$ $\angle BEC \neq 90°$, $\therefore$ G is not the orthocentre of ΔABC.

Unit 9 Areas & volumes (3): Pyramids, cones & spheres

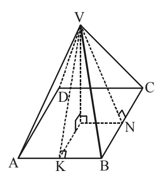


1. (a) $VM = \sqrt{25^2 - (\frac{14}{2})^2} = \sqrt{576} = 24$ cm

Total surface area = $14^2 + \frac{14 \times 24}{2} \times 4 = 196 + 672 = 868$ cm^2

(b) Height = $\sqrt{24^2 - (\frac{14}{2})^2} = \sqrt{527} = 23.0$ cm

Volume = $\frac{1}{3} \times 14^2 \times \sqrt{527} = 1\,499.8$ cm^3



2. (a) Volume = $\frac{1}{3} \times 20 \times 10 \times 12 = 800$ cm^3

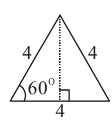
(b) $VN = \sqrt{12^2 + (\frac{10}{2})^2} = \sqrt{169} = 13$ cm

$VK = \sqrt{12^2 + (\frac{20}{2})^2} = \sqrt{244} = 2\sqrt{61} = 15.6$ cm

(c) Total surface area = $(\frac{13 \times 20}{2} + \frac{2\sqrt{61} \times 10}{2}) \times 2 + 10 \times 20 = 616.2$ cm^2

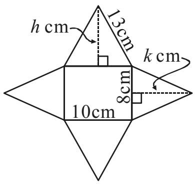
3. Diagonals of a rhombus are perpendicular to and bisect each other,

$\therefore$ base area = $(\frac{1}{2} \times 24 \times \frac{10}{2}) \times 2 = 120$ cm^2 $\therefore$ Volume = $\frac{1}{3} \times 120 \times 16 = 640$ cm^3



4. Let h cm be the height of the base. $\frac{h}{4} = \sin 60° = \frac{\sqrt{3}}{2}$, $h = 2\sqrt{3}$

$\therefore$ Total surface area = $\frac{4 \times 2\sqrt{3}}{2} \times 4 = 16\sqrt{3} = 27.7$ cm^2



5. (a) Let h cm and k cm be the two slant heights.

$h=\sqrt{13^2-(\frac{10}{2})^2}=\sqrt{144}=12$, $k=\sqrt{13^2-(\frac{8}{2})^2}=\sqrt{153}$

$\therefore$ Total surface area $=(\frac{12\times10}{2}+\frac{\sqrt{153}\times8}{2})\times2+10\times8$
$= 299.0\text{ cm}^2$

(b) Height of the pyramid $=\sqrt{12^2-(\frac{8}{2})^2}=\sqrt{128}$ cm

$\therefore$ Volume $=\frac{1}{3}\times10\times8\times\sqrt{128}=301.7\text{ cm}^3$

6. (a) Let y cm be the side of the base. $\frac{1}{3}\times y^2\times15=2\,000$, $y^2=400$,

$y=\sqrt{400}=20$. $\therefore$ Area of the base $=20^2=400\text{ cm}^2$

(b) Slant height $=\sqrt{15^2+(\frac{20}{2})^2}=\sqrt{325}$ cm

$\therefore$ Length of slant edge $=\sqrt{(\sqrt{325})^2+(\frac{20}{2})^2}=\sqrt{425}=20.6$ cm

(c) Total surface area $=\frac{\sqrt{325}\times20}{2}\times4+20^2=1121.1\text{ cm}^2$

7. Volume $=\frac{4}{3}\pi(\frac{7}{2})^3=\frac{343\pi}{6}=179.6\text{ cm}^3$

8. Volume $=\frac{4}{3}\pi(11)^3\times\frac{1}{2}=\frac{2662\pi}{3}\text{ cm}^3$

Total surface area $=4\pi(11)^2\times\frac{1}{2}+\pi(11)^2=242\pi+121\pi=363\pi\text{ cm}^2$

9. Let r cm be the radius. $4\pi r^2=100$, $r^2=\frac{25}{\pi}$, $r=\sqrt{\frac{25}{\pi}}$

$\therefore$ Diameter $=2\times\sqrt{\frac{25}{\pi}}=5.64$ cm

10. Let r cm be the radius. $4\pi r^2=320$, $r=\sqrt{\frac{80}{\pi}}$

$\therefore$ Volume $=\frac{4}{3}\pi\left(\sqrt{\frac{80}{\pi}}\right)^3=538.3\text{ cm}^3$

11. (a) Let r cm be the new radius. $\frac{4}{3}\pi r^3=\frac{4}{3}\pi(3^3+5^3+7^3)$,

$r^3=3^3+5^3+7^3=495$, $\therefore$ $r=\sqrt[3]{495}=7.91$.
Ans. The radius is 7.91 cm.

(b) Original surface area $=4\pi(3^2+5^2+7^2)=332\pi\text{ cm}^2$,
new surface area $=4\pi(\sqrt[3]{495})^2=250.3\pi\text{ cm}^2$.

$\therefore$ % change in surface area $=\frac{250.3\pi-332\pi}{332\pi}\times100\%=-24.6\%$

Ans. The surface area decreases by 24.6%.

12. Let R and r be the radii of original and new spheres respectively.

$\frac{4}{3}\pi R^3 = \frac{4}{3}\pi r^3 \times 8$, $R^3 = 8r^3$, $R = \sqrt[3]{8r^3} = 2r$, $r = \frac{R}{2}$.

$\therefore$ % change in total surface area = $\frac{4\pi r^2 \times 8 - 4\pi R^2}{4\pi R^2} \times 100\%$

$= \frac{32\pi(\frac{R}{2})^2 - 4\pi R^2}{4\pi R^2} \times 100\% = \frac{4\pi R^2}{4\pi R^2} \times 100\% = 100\%$ (increase)

13. External and internal radii are $\frac{15}{2}$ cm and $\frac{15-2\times 2}{2} = \frac{11}{2}$ cm respectively.

Volume of hollow sphere = $\frac{4}{3}\pi[(\frac{15}{2})^3 - (\frac{11}{2})^3] = 1\,070.235$ cm^3

$\therefore$ Weight = 1 070.235 × 150 = 160 535 g

14. Let h cm be the rise in water level. $\pi(6)^2 h = \frac{4}{3}\pi(\frac{3}{2})^3 \times 10$, $36h = 45$, $h = 1.25$

Ans. The rise in water level is 1.25 cm.

15. (a) Volume = $\frac{1}{3}\pi(6)^2(7) = 84\pi = 263.9$ cm^3. Slant edge = $\sqrt{6^2 + 7^2} = \sqrt{85}$ cm

$\therefore$ Total surface area = $\pi(6)(\sqrt{85}) + \pi(6)^2 = 286.9$ cm^2

(b) Height = $\sqrt{13^2 - 5^2} = \sqrt{144} = 12$ cm

$\therefore$ Volume = $\frac{1}{3}\pi(5)^2(12) = 100\pi = 314.2$ cm^3

Total surface area = $\pi(5)(13) + \pi(5)^2 = 90\pi = 282.7$ cm^2

16. Let r cm be the base radius, $2\pi r = 30$, $r = \frac{15}{\pi}$, $\therefore$ Slant edge = $\sqrt{\left(\frac{15}{\pi}\right)^2 + 20^2} = 20.562$,

$\therefore$ Curved surface area = $\pi(\frac{15}{\pi})(20.562) = 308.4$ cm^2

17. Let ℓ cm be the length of slant edge, $\pi(5)(\ell) = 40\pi$, $\ell = 8$,

$\therefore$ Height = $\sqrt{8^2 - 5^2} = \sqrt{39}$. $\therefore$ Volume = $\frac{1}{3}\pi(5)^2(\sqrt{39}) = 163.5$ cm^3

18. Let r cm be the base radius, $\frac{1}{3}\pi(r)^2(12) = \pi(6)^2(10)$, $4r^2 = 360$, $r^2 = 90$,

$\therefore$ $r = \sqrt{90} = 9.5$. *Ans. The base radius is 9.5 cm.*

19. Let r cm be the base radius, then the height is $2r$ cm.

$\frac{1}{3}\pi r^2(2r) = 1152\pi$, $r = \sqrt[3]{1728} = 12$, $\therefore$ Slant edge = $\sqrt{12^2 + (2\times 12)^2} = \sqrt{720} = 12\sqrt{5}$

$\therefore$ Curved surface area = $\pi(12)(12\sqrt{5}) = 144\sqrt{5}\pi$ cm^2

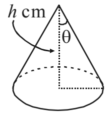
20. $\theta = 60° \div 2 = 30°$. Let h cm be the height.

$\tan 30° = \frac{15}{h}$, $h = \frac{15}{\tan 30°} = 15\sqrt{3}$

$\therefore$ Slant edge = $\sqrt{15^2 + (15\sqrt{3})^2} = 30$ cm,

$\therefore$ Total surface area = $\pi(15)(30) + \pi(15)^2 = 675\pi = 2120.6$ cm^2

Volume = $\frac{1}{3}\pi(15)^2(15\sqrt{3}) = 1125\sqrt{3}\pi = 6121.6$ cm^3



21. Let r cm be the base radius of the cone.

$2\pi r = 2\pi(\frac{14}{2}) \times \frac{1}{2}$, $r = \frac{7}{2}$, $\therefore$ height = $\sqrt{(\frac{14}{2})^2 - (\frac{7}{2})^2} = \frac{7\sqrt{3}}{2}$

$\therefore$ Volume = $\frac{1}{3}\pi(\frac{7}{2})^2(\frac{7\sqrt{3}}{2}) = \frac{343\sqrt{3}\pi}{24} = 77.8$ cm^3

22. (a) Curved surface area = $\pi(9)^2 \times \frac{280°}{360°} = 63\pi = 197.9$ cm^2

(b) Let r cm be the base radius of the cone.

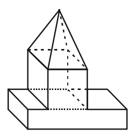
$\pi r(9) = 63\pi$, $r = 7$, $\therefore$ height = $\sqrt{9^2 - 7^2} = 4\sqrt{2}$

$\therefore$ Volume = $\frac{1}{3}\pi(7)^2(4\sqrt{2}) = 290.3$ cm^3

23. Volume = $5(1)(1) + (1)(5-2-1)(3-1) + \frac{1}{3}(1)(5-2-1)(2) = 5 + 4 + \frac{4}{3} = 10\frac{1}{3}$ cm^3

24. (a) Volume = $\frac{1}{3}\pi(5)^2(12) + \pi(5)^2(25-12-5) + \frac{4}{3}\pi(5)^3 \times \frac{1}{2}$

$= 100\pi + 200\pi + \frac{250\pi}{3} = \frac{1150\pi}{3} = 1\,204.3$ cm^3

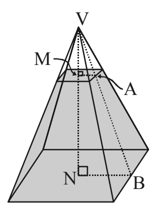


(b) Slant edge of the cone = $\sqrt{5^2 + 12^2} = \sqrt{169} = 13$ cm,
height of cylinder = 25 – 12 – 5 = 8 cm

$\therefore$ Total sruface area = $\pi(5)(13) + 2\pi(5)(8) + 4\pi(5)^2 \times \frac{1}{2} = 195\pi = 612.6$ cm^2

25. (a) $\because$ ΔVMA~ΔVNB, $\therefore$ $\frac{MA}{NB} = \frac{VM}{VN}$, $\frac{MA}{5} = \frac{4}{4+8}$, MA = $\frac{5}{3}$

$\therefore$ Length of the base of the small pyramid = $\frac{5}{3} \times 2 = \frac{10}{3}$ cm



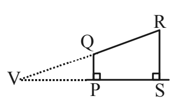
$\therefore$ Volume of frustum

$= \frac{1}{3}\pi[(10)^2(12) - (\frac{10}{3})^2(4)] = \frac{10400\pi}{27} = 1\,210.1$ cm^3

(b) VB = $\sqrt{12^2 + 5^2} = 13$, VA = $\sqrt{4^2 + (\frac{5}{3})^2} = \sqrt{\frac{169}{9}} = \frac{13}{3}$

$\therefore$ Total surface area = $10^2 + (\frac{10}{3})^2 + 4 \times \frac{1}{2} \times (10 \times 13 - \frac{10}{3} \times \frac{13}{3}) = 342.2$ cm^2

26. (a) $\because$ ΔVQP~ΔVRS, $\therefore$ $\frac{VP}{VS} = \frac{QP}{RS}$, $\frac{VP}{VP+10} = \frac{3}{6} = \frac{1}{2}$, VP = 10



$\therefore$ VQ = $\sqrt{3^2 + 10^2} = \sqrt{109}$; VR = $\sqrt{6^2 + (10+10)^2} = \sqrt{436}$

$\therefore$ Lateral surface area = $\pi(6)(\sqrt{436}) - \pi(3)(\sqrt{109}) = 295.2$ cm^2

(b) Volume = $\frac{1}{3}\pi[(6)^2(10+10)-(3)^2(10)]=210\pi$ = 659.7 cm^3

27. The space is one-eighths of a sphere with radius 1 m.

$\therefore$ Volume of space = $\frac{4}{3}\pi(1)^3\times\frac{1}{8}=\frac{\pi}{6}$ = 0.524 m^3

28. (a) Slant height = $\sqrt{(2.6x)^2-(\frac{2x}{2})^2}=\sqrt{5.76x^2}$ = 2.4x cm

(b) $\frac{(2x)(2.4x)}{2}\times4+(2x)^2=1360$, $13.6x^2=1360$, $x^2=100$, $\therefore$ $x=\sqrt{100}$ = 10

(c) Height of pyramid = $\sqrt{(2.4x)^2-x^2}=\sqrt{24^2-10^2}=\sqrt{476}$

$\therefore$ Volume = $\frac{1}{3}(20)^2\sqrt{476}$ = 2 909.0 cm^3

29. AD = AH = AB = $\sqrt[3]{a}$ cm.

$\therefore$ Volume of tetrahedron BADH

$=\frac{1}{3}\times$ area of ΔABH $\times$ AD = $\frac{1}{3}(\frac{\sqrt[3]{a}\times\sqrt[3]{a}}{2})(\sqrt[3]{a})=\frac{1}{6}(\sqrt[3]{a})^3=\frac{a}{6}$ cm^3

$\therefore$ Volume of remaining solid = $a-\frac{a}{6}=\frac{5a}{6}$ cm^3

30. (a) Capacity = $\frac{4}{3}\pi(1)^3\times\frac{1}{2}+\pi(1)^2(10-1)=\frac{29}{3}\pi$ = 30.4 cm^3

(b) Let h cm be the height of water level of the cylindrical part.

$\pi(1)^2(h)+\frac{4}{3}\pi(1)^3\times\frac{1}{2}=\frac{29}{3}\pi\times\frac{2}{3}$, $h+\frac{2}{3}=\frac{58}{9}$, $h=\frac{52}{9}$

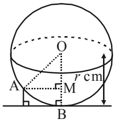
$\therefore$ Total area of wet surface = $2\pi(1)(\frac{52}{9})+4\pi(1)^2\times\frac{1}{2}=\frac{104\pi}{9}+2\pi=\frac{122\pi}{9}$ = 42.6 cm^2

31. Let r cm be the radius.

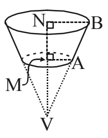
$AM^2+OM^2=OA^2$, $3^2+(r-1)^2=r^2$, $9+r^2-2r+1=r^2$, $r=5$.

Surface area = $4\pi(5)^2$ = 100π = 314.2 cm^2

Volume = $\frac{4}{3}\pi(5)^3=\frac{500\pi}{3}$ = 523.6 cm^3



32. (a) Capacity = $\pi(4)^2(5-4)+\frac{4}{3}\pi(4)^3\times\frac{1}{2}=\frac{176\pi}{3}$ cm^3



(b) $\because$ ΔVAM ~ ΔVBN, $\therefore$ $\frac{VM}{VM+6}=\frac{4}{6}=\frac{2}{3}$, VM = 12,

$\therefore$ VN = 12 + 6 = 18.

$\therefore$ Volume of mould = $\frac{1}{3}\pi[(6)^2(18)-(4)^2(12)]-\frac{176\pi}{3}=\frac{280\pi}{3}$ cm^3

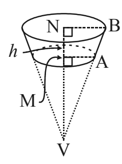
$\therefore$ Weight = $\frac{280\pi}{3}\times0.8$ = 234.6 g

33. (a) $\because$ ΔVAM~ΔVBN, $\therefore$ $\frac{VM}{VN}=\frac{AM}{BN}$, $\frac{VM}{VM+h}=\frac{6}{8}=\frac{3}{4}$,

VM = 3h, $\therefore$ VN = 3h + h = 4h

$\therefore$ $\frac{1}{3}\pi[(8)^2(4h)-(6)^2(3h)]=148\pi$, $\frac{148\pi h}{3}=148\pi$, $h=3$

Ans. Height of the frustum is 3 cm.



(b) VA = $\sqrt{6^2+(3\times3)^2}=\sqrt{117}$, VB = $\sqrt{8^2+(4\times3)^2}=\sqrt{208}$,

$\therefore$ Total surface area = $\pi(8)(\sqrt{208})-\pi(6)(\sqrt{117})+\pi(6)^2+\pi(8)^2$ = 472.7 cm^2

34. (a) Let r cm be the radius.

Volume$_{\text{hemisphere}}$: Volume$_{\text{cone}}$ = $\frac{4}{3}\pi r^3\times\frac{1}{2}$: $\frac{1}{3}\pi r^2(r)=\frac{2}{3}:\frac{1}{3}=2:1$

(b) Let base area of the tank = A cm^2, rise in water level = h cm.

$A(6):A(h)=2:1$, $\therefore$ rise in water level = $6\times\frac{1}{2}$ = 3 cm

35. (a) $\frac{1}{3}\pi(\frac{5}{2})^2(x)+\pi(\frac{5}{2})^2(8-\frac{5}{2})+\frac{4}{3}\pi(\frac{5}{2})^3\times\frac{1}{2}=\pi(\frac{5}{2})^2(x+8)(1-20\%)$,

$\frac{25x}{12}+\frac{275}{8}+\frac{125}{12}=\frac{25}{4}\times\frac{4}{5}(x+8)$, $\frac{5x}{12}+\frac{55}{8}+\frac{25}{12}=x+8$,

$10x+165+50=24x+192$, $23=14x$, $\therefore$ $x=\frac{23}{14}$

(b) Slant edge fo circular cone = $\sqrt{\left(\frac{5}{2}\right)^2+\left(\frac{23}{14}\right)^2}$ = 2.9915 cm,

$\therefore$ Total surface area = $\pi\left(\frac{5}{2}\right)(2.9915)+\pi(5)\left(8-\frac{5}{2}\right)+4\pi\left(\frac{5}{2}\right)^2\times\frac{1}{2}$ = 149.2 cm^2

$\therefore$ Cost = 149.2 × 0.4 = \$59.7

36. (a) Let h_1 cm and h_2 cm be the heights of cones A and (A+B) respectively.

$\frac{h_1}{30}=\frac{5}{15}$, $h_1=10$; $\frac{h_2}{30}=\frac{12}{15}$, $h_2=24$.

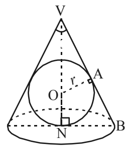
$\therefore$ Volume of frustum B = $\frac{1}{3}\pi[(12)^2(24)-(5)^2(10)]$ = 3 357.3 cm^3.

(b) Lateral surface area of frustum B = $\pi(12)(\sqrt{12^2+24^2})-\pi(5)(\sqrt{5^2+10^2})$

$=\pi(12\times12\sqrt{5}-5\times5\sqrt{5})=119\sqrt{5}\pi$ cm^2

(c) Total surface area = $\pi(15)(\sqrt{30^2+15^2})-119\sqrt{5}\pi+\pi(15^2+12^2-5^2)$ = 1 825.3 cm^2

37. (a) $\angle$OVA = 60° ÷ 2 = 30°.



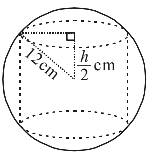
In ΔOVA, $\sin 30°=\frac{OA}{OV}$, $\frac{1}{2}=\frac{r}{OV}$, OV = $2r$.

Height of cone = OV + ON = $2r+r=3r$

(b) In ΔVNB, $\tan 30°=\frac{BN}{VN}$, $\frac{1}{\sqrt{3}}=\frac{BN}{3r}$, BN = $\sqrt{3}r$

$\therefore$ Volume$_{\text{cone}}$: Volume$_{\text{sphere}}$ = $\frac{1}{3}\pi(\sqrt{3}r)^2(3r):\frac{4}{3}\pi r^3$ = $3\pi r^3:\frac{4}{3}\pi r^3$ = 9 : 4

38. (a) Base radius of cylinder = $\sqrt{12^2-(\frac{h}{2})^2}=\sqrt{144-\frac{h^2}{4}}$ cm



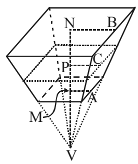
$\therefore$ $V=\pi\left(\sqrt{144-\frac{h^2}{4}}\right)^2(h)=\pi h\left(144-\frac{h^2}{4}\right)=\frac{\pi h}{4}(576-h^2)$

(b) $\frac{2h}{3} = 12$, $h = 18$. $\therefore$ $V = \frac{18\pi}{4}(576 - 18^2) = 1\,134\pi$

$\therefore$ $\text{Volume}_{\text{cylinder}} : \text{Volume}_{\text{wood remained}}$

$= 1134\pi : \left[\frac{4}{3}\pi(12)^3 - 1134\pi\right] = 1134\pi : 1170\pi = 63 : 65$

39. (a) $\because$ $\Delta VMA \sim \Delta VNB$, $\therefore$ $\frac{VM}{VN} = \frac{AM}{BN}$,



$\frac{VM}{VM+12} = \frac{4}{8}$, $2VM = VM + 12$, $VM = 12$

$\therefore$ VN = 12 + 12 = 24.

Volume of frustum = $\frac{1}{3}(16)^2(24) - \frac{1}{3}(8)^2(12) = 1\,792 \text{ cm}^3$

(b) $\because$ $\Delta VMA \sim \Delta VPC$, $\therefore$ $\frac{VM}{VP} = \frac{AM}{CP}$, $\frac{12}{12+9} = \frac{4}{CP}$, CP = 7.

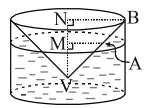
$\therefore$ Length of side of water surface = 7 × 2 = 14 cm

$\therefore$ Volume of water = $\frac{1}{3}(14)^2(21) - \frac{1}{3}(8)^2(12) = 1116 \text{ cm}^3$

(c) Water flows from the pipe to the frustum in 1 second = $\pi\,(0.8)^2\,(6)$ cm^3

$\therefore$ Time taken = $1116 \div \pi\,(0.8)^2\,(6) = 92.5$ seconds.

40. MN = 12 – 9 = 3 cm, VN = 9 cm, $\therefore$ VM = 9 – 3 = 6 cm



$\because$ $\Delta VMA \sim \Delta VNB$, $\therefore$ $\frac{AM}{8} = \frac{VM}{VN} = \frac{6}{9}$, $AM = \frac{16}{3}$

Let h cm be the new water level.

$\pi(8)^2 h = \pi(8)^2(9) - \frac{1}{3}\pi(AM)^2(VM)$,

$(8)^2 h = (8)^2(9) - \frac{1}{3}\left(\frac{16}{3}\right)^2(6)$, $64h = 576 - \frac{512}{9}$, $\therefore$ $h = \frac{73}{9} = 8.11$

Ans. The new water level is 8.11 cm.

41. (a) $AB^2 + BC^2 = (3^2 + 4^2) = 25 \text{ cm}^2$, $AC^2 = 5^2 = 25 \text{ cm}^2$

$\because$ $AB^2 + BC^2 = AC^2$, $\therefore$ $\angle ABC = 90°$ (Converse of Pyth. thm.)

$\therefore$ ΔABC is a right-angles triangle.

(b) Volume of pyramid VABC = $\frac{1}{3}\left(\frac{3\times 4}{2}\right)(6) = 12 \text{ cm}^3$

(c) Let BD = h cm.

$\frac{1}{3}\left(2\sqrt{29}\right)(h) = 12$, $h = \frac{12}{\sqrt{29}} = \frac{12\sqrt{29}}{29}$, $\therefore$ BD = $\frac{12\sqrt{29}}{29}$ cm.

42. (a) Let r cm be the base radius.

$\frac{1}{3}(\pi r^2)(12) = 100\pi$, $r^2 = 25$, $r = 5$, $\therefore$ base radius is 5 cm.

(b) (i) Slant height of the paper cup = $\sqrt{5^2 + 12^2} = 13$ cm

$\therefore$ Radius of the sector = 13 cm.

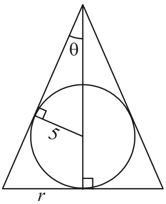
(ii) Let θ be the angle of the sector.

$$2\pi(13)\left(\frac{\theta}{360°}\right)=2\pi(5),\quad \frac{\theta}{360°}=\frac{5}{13},\quad \theta=138.5°\ (1\text{ d.p.})$$

Ans. The angle of the sector is 138.5°.

43\. (a) VA = VB (given), ∠VNA = ∠VNB (base ∠s, isos. Δ) ,
VN = VN (common side), ∴ ΔVNA ≅ ΔVNB (RHS) ,
∴ ∠AVN = ∠BVN (corr. ∠s, ≅Δs), CG = DG (radii) ,
∠VCG = VDG = 90° (prop. of ∠ bisector)

(b) (i) radius of the sphere = $\sqrt{4^2+\left(\frac{6}{2}\right)^2}$ = 5 cm



(ii) Let r cm be the base radius of the cone.
Let 2θ be the vertical angle of the cone.

$$\frac{r}{15}=\tan\theta=\frac{5}{\sqrt{(15-5)^2-5^2}},\quad \therefore\ r=\frac{75}{\sqrt{75}}=\sqrt{75}=5\sqrt{3}$$

∴ Radius of the cone is $5\sqrt{3}$ cm.

(iii) Volume of the cone = $\frac{1}{3}\pi(5\sqrt{3})^2(15)$ = 375π cm^3

Volume of the sphere = $\frac{4}{3}\pi(5)^2$ = $\frac{500}{3}\pi$ cm^3

The required ratio = $375\pi : \frac{500}{3}\pi$ = 1125 : 500 = 9 : 4

Unit 10 Area and volume of similar solids

1. Ratio of heights = 1.2 : 0.9 = 4 : 3, ∴ ratio of volumes = $4^3 : 3^3$ = 64 : 27
2. Ratio of capacities = 162 : 750 = 27: 125
   ∴ Ratio of surface areas = $(\sqrt[3]{27})^2 : (\sqrt[3]{125})^2$ = 9 : 25
3. Ratio of diameters = (10 + 2 × 2) : 10 = 7 : 5
   ∴ External surface area : internal surface area = $7^2 : 5^2$ = 49 : 25
4. Ratio of heights = 9 : 12 = 3: 4, ratio of volumes = $3^3 : 4^3$ = 27 : 64
   ∴ Weight of bigger pyramid = 270 × $\frac{64}{27}$ = 640g
5. Ratio of volumes = 1 kg : 125 g = 1000 : 125 = 8 : 1
   Ratio of surface areas = $(\sqrt[3]{8})^2 : (\sqrt[3]{1})^2$ = 4 : 1
   ∴ Cost of painting the larger solid = 18 × 4 = $72
6. (a) Let the original radius and area be r_1, A_1; those of the new ones be r_2, A_2.

$$\left(\frac{r_2}{r_1}\right)^2=\frac{A_2}{A_1}=\frac{A_1(1+69\%)}{A_1}=\frac{1.69}{1},\quad \frac{r_2}{r_1}=\sqrt{\frac{1.69}{1}}=\frac{1.3}{1},$$

∴ Percentage increase in radius = $\frac{1.3r_1-r_1}{r_1}\times100\%$ = 30%

(b) $\dfrac{V_2}{V_1}=\left(\dfrac{r_2}{r_1}\right)^3=\left(\dfrac{1.3}{1}\right)^3=\dfrac{2.197}{1}$,

$\therefore$ Percentage increase in volume = $\dfrac{2.197V_1-V_1}{V_1}\times 100\%$ = 119.7%

7. Let the original length, area and volume be r_1, A_1, V_1; those of the new ones be r_2, A_2, V_2.

$\left(\dfrac{r_2}{r_1}\right)^3=\dfrac{V_2}{V_1}=\dfrac{V_1(1-27.1\%)}{V_1}=\dfrac{0.729}{1}$, $\dfrac{r_2}{r_1}=\sqrt[3]{\dfrac{0.729}{1}}=\dfrac{0.9}{1}$, $\dfrac{A_2}{A_1}=\left(\dfrac{r_2}{r_1}\right)^2=\left(\dfrac{0.9}{1}\right)^2=\dfrac{0.81}{1}$,

$\therefore$ Percentage change in area = $\dfrac{0.81A_1-A_1}{A_1}\times 100\% = -19\%$ (decrease)

8. $\dfrac{\text{Vol. of small pendulum}}{\text{Vol. of big pendulum}}=\dfrac{1}{4}$. Let h cm be the height of a small pendulum.

$\left(\dfrac{h}{16}\right)^3=\dfrac{1}{4}$, $\dfrac{h}{16}=\sqrt[3]{\dfrac{1}{4}}$, $h = 10.1$ *Ans. The height of a small pendulum is 10.1 cm.*

9. $V_A : V_{A+B} : V_{A+B+C} = y^3 : (2y)^3 : (3y)^3 = 1 : 8 : 27$

$\therefore$ $V_A : V_B : V_C = 1 : (8-1) : (27-8) = 1 : 7 : 19$

10. PQ : PR : PS = 2 : (2+1) : (2+1+3) = 2 : 3 : 6,

$\therefore$ Area of circle I : area of circle II : area of circle III = $2^2 : 3^2 : 6^2 = 4 : 9 : 36$

11. $\dfrac{\text{Vol. of small pyramid}}{\text{Vol. of big pyramid}}=\left(\sqrt{\dfrac{49}{64}}\right)^3=\left(\dfrac{7}{8}\right)^3=\dfrac{343}{512}$; $\dfrac{\text{weight of frustum}}{\text{weight of big pyramid}}=\dfrac{512-343}{512}=\dfrac{169}{512}$

$\therefore$ Weight of frustum = $2\times\dfrac{169}{256}$ = 1.32 kg

12. $\left(\dfrac{AQ}{AB}\right)^3=\dfrac{216}{216+513}=\dfrac{216}{729}=\dfrac{8}{27}$, $\dfrac{AQ}{AB}=\sqrt[3]{\dfrac{8}{27}}=\dfrac{2}{3}$, $\therefore$ AQ : QB = 2 : (3 – 2) = 2 : 1

13. $\dfrac{\text{Old volume}}{\text{New volume}}=\dfrac{60}{60+420}=\dfrac{1}{8}$; $\dfrac{\text{Old wet area}}{\text{New wet area}}=\left(\sqrt[3]{\dfrac{1}{8}}\right)^2=\left(\dfrac{1}{2}\right)^2=\dfrac{1}{4}$

$\therefore$ Increase in wet surface area = 25 × 4 – 25 = 75cm^2

14. (a) $\dfrac{\text{Old volume}}{\text{New volume}}=\left(\dfrac{16}{24}\right)^3=\left(\dfrac{2}{3}\right)^3=\dfrac{8}{27}$

$\therefore$ Volume of water added = $400\times\dfrac{27}{8}-400 = 950\ \text{cm}^3$

(b) $\dfrac{\text{Old wet surface area}}{\text{New wet surface area}}=\left(\dfrac{16}{24}\right)^2=\left(\dfrac{2}{3}\right)^2=\dfrac{4}{9}$

$\therefore$ Percentage change in wet surface area = $\dfrac{9-4}{4}\times 100\%$ = 125% (increase)

15. PT = 3TR, $\therefore$ PR = 3TR + TR = 4TR; QR = 3TR, $\therefore$ QR = 3SR + SR = 4SR

In ΔPQR and ΔTSR, ∠R = ∠R (common), $\dfrac{PR}{TR}=\dfrac{4TR}{TR}=4$, $\dfrac{QR}{SR}=\dfrac{4SR}{SR}=4$,

$\therefore$ ΔPQR ~ ΔTSR (ratio of 2 sides, inc. ∠)

$\therefore\ \dfrac{\text{Area of }\Delta\text{STR}}{\text{Area of }\Delta\text{PQR}}=\left(\dfrac{\text{TR}}{\text{PR}}\right)^2=\left(\dfrac{1}{4}\right)^2=\dfrac{1}{16},\quad \dfrac{\text{Area of }\Delta\text{STR}}{\text{Area of }P\text{QST}}=\dfrac{1}{16-1}=\dfrac{1}{15}$

16. $\because\ \Delta\text{AEF}\sim\Delta\text{ADB}$ (AAA), $\therefore\ \dfrac{\text{Area of }\Delta\text{AEF}}{\text{Area of }\Delta\text{ADB}}=\left(\dfrac{\text{AE}}{\text{AD}}\right)^2=\left(\dfrac{2\text{ED}}{\text{AE}+\text{ED}}\right)^2=(\dfrac{2\text{ED}}{3\text{ED}})^2=\dfrac{4}{9}$

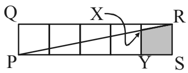
$\because\ \Delta\text{CGH}\sim\Delta\text{CBD}$ (AAA), $\therefore\ \dfrac{\text{Area of }\Delta\text{CGH}}{\text{Area of }\Delta\text{CBD}}=\left(\dfrac{\text{GC}}{\text{BC}}\right)^2=\left(\dfrac{\text{GC}}{\text{BG}+\text{GC}}\right)^2=\left(\dfrac{\text{GC}}{2\text{GC}}\right)^2=\dfrac{1}{4}$

Since area of ΔADB = area of ΔCBD,

$\therefore$ percentage of ABCD shaded = $\dfrac{4}{9}\times 50\%+\dfrac{1}{4}\times 50\%=34.7\%$

17. $\because\ \Delta\text{PXY}\sim\Delta\text{PRS}$ (AAA),

$\therefore\ \dfrac{\text{Area of }\Delta\text{PXY}}{\text{Area of }\Delta\text{PRS}}=\left(\dfrac{\text{PY}}{\text{PS}}\right)^2=\left(\dfrac{4}{5}\right)^2=\dfrac{16}{25}$



Area of ΔPRS = $\dfrac{6\times 5}{2}$ = 15cm^2, $\therefore$ shaded area = 15 – 15× $\dfrac{16}{25}$ = 5.4 cm^2

18. (a) $\because\ \Delta\text{ABC}\sim\Delta\text{ADE}$ (A.A.A.), $\therefore\ \left(\dfrac{\text{BC}}{\text{DE}}\right)^2=\dfrac{\text{Area of }\Delta\text{ABC}}{\text{Area of }\Delta\text{ADE}}=\dfrac{50}{50+112}=\dfrac{25}{81}$,

$\therefore$ BC : DE = $\sqrt{25}:\sqrt{81}$ = 5 : 9

(b) $\because$ AC : AE = BC : DE = 5 : 9, $\therefore$ AC : CE = 5 : (9 – 5) = 5 : 4,

AE : AG = (5+4) : (5+4+4) = 9 : 13. $\because\ \Delta\text{ADE}\sim\Delta\text{AFG}$ (A.A.A.),

$\therefore$ DE : FG = AE : AG = 9 : 13. $\dfrac{\text{Area of }\Delta\text{ADE}}{\text{Area of }\Delta\text{AFG}}=\left(\dfrac{\text{DE}}{\text{FG}}\right)^2=\left(\dfrac{9}{13}\right)^2=\dfrac{81}{169}$

$\therefore$ Area of ΔAFG = (50 + 112) × $\dfrac{169}{81}$ = 338 cm^2

19. Let A_1, A_2, A_3 be the curved surface areas of portions I, II, III respectively.

$\dfrac{A_1}{A_1+A_2}=\left(\dfrac{2h}{3h}\right)^2=\dfrac{4}{9},\quad 9A_1=4(A_1+36),\quad 5A_1=144,\quad A_1=28.8$

$\dfrac{A_1}{A_1+A_2+A_3}=\left(\dfrac{2h}{4h}\right)^2=\left(\dfrac{1}{2}\right)^2=\dfrac{1}{4},$

$\therefore\ \dfrac{28.8}{28.8+36+A_3}=\dfrac{1}{4},\quad 115.2=28.8+36+A_3,\quad A_3=50.4.$

Ans. The curved surface areas of portions I and III are 28.8 cm^2 and 50.4 cm^2 respectively.

20. Vol. of B : Vol. of C = 3 : 2, Vol. of A : Vol. of B = $(\sqrt{4})^3:(\sqrt{1})^3$ = 8 : 1 = 24 : 3

$\therefore$ Vol. of A : Vol. of B : Vol. of C = 24 : 3 :2,

$\therefore$ Vol. of A : Vol. of C = 24 : 2 = 12 : 1

21. (a) Volume of M : volume of N = $1^3:2^3$ = 1: 8

(b) Let the radius of A be $2x$ cm, radius of B be $3x$ cm, water risen in cylinder B be h cm.

$\dfrac{\text{Vol. of N}}{\text{Vol. of M}}=\dfrac{\text{Vol. of water risen in B}}{\text{Vol. of water risen in A}},\quad \therefore\ \dfrac{\pi(3x)^2h}{\pi(2x)^2(6)}=\dfrac{8}{1},\quad \dfrac{9h}{24}=8,\quad h=\dfrac{64}{3}$

Ans. The rise in water level in B is $21\frac{1}{3}$ cm.

22. (a) Vol. of space : Vol. of vessel = $(12-4)^3 : 12^3 = 8 : 27$

$\therefore$ Vol. of water : Vol. of vessel = $(27-8) : 27 = 19 : 27$

Let h cm be the depth of water now. $\frac{h}{12} = \sqrt[3]{\frac{19}{27}}$, $h = 12\times\sqrt[3]{\frac{19}{27}} = 10.7$

Ans. The depth of water now is 10.7 cm.

(b) Let the curved surface area of the vessel, the curved area of the original space, the original wet lateral surface and the new wet surface be x, A_{space}, $A_{original}$ and A_{new}.

$\frac{A_{space}}{x} = \left(\sqrt[3]{\frac{8}{27}}\right)^2 = \frac{4}{9}$, $\therefore$ $\frac{A_{original}}{x} = \frac{9-4}{9} = \frac{5}{9}$, $A_{original} = \frac{5}{9}x$

$\frac{A_{new}}{x} = \left(\sqrt[3]{\frac{19}{27}}\right)^2 = \frac{(\sqrt[3]{19})^2}{9}$, $\therefore$ $A_{new} = \frac{(\sqrt[3]{19})^2}{9}x$

$\frac{A_{new}}{A_{originald}} = \frac{(\sqrt[3]{19})^2}{9} \div \frac{5}{9} = \frac{(\sqrt[3]{19})^2}{5}$.

$\therefore$ Percentage change in curved wet surface areas $= \frac{(\sqrt[3]{19})^2 - 5}{5} \times 100\% = 42.4\%$ (increase)

23. (a) Volume of water = $\frac{4}{3}\pi(3)^3 \times \frac{1}{2} + \pi(3)^2(19-3) = 18\pi + 144\pi = 162\pi \text{ cm}^3$

(b) Let V cm^3 be the volume of oil.

$\frac{162\pi}{V+162\pi} = \left(\frac{3}{3+1}\right)^3 = \frac{27}{64}$, $10\,368\pi = 27V + 4\,374\pi$, $V = 222\pi$

Let h cm be the depth of oil in the glass-tube.

$\frac{4}{3}\pi(3)^3 \times \frac{1}{2} + \pi(3)^2(h-3) = 222\pi$, $18\pi + 9\pi(h-3) = 222\pi$, $9(h-3) = 204$, $h = 25.7$

Ans. The depth of oil in the glass-tube is 25.7 cm.

24. (a) $\frac{\text{Area of }\Delta CDE}{\text{Area of }\Delta BDE} = \frac{CE}{BE}$ (Δs with same base), area of ΔCDE = $\frac{1}{2} \times 152 = 76 \text{ cm}^2$

(b) $\frac{\text{Area of }\Delta BMN}{\text{Area of }\Delta BAD} = \left(\frac{BM}{BA}\right)^2 = \left(\frac{BM}{2BM}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$,

$\therefore$ $\frac{\text{Area of }\Delta BMN}{\text{Area of ADNM}} = \frac{1}{4-1} = \frac{1}{3}$, area of ΔBMN = $\frac{1}{3} \times 360 = 120 \text{ cm}^2$

(c) $\frac{CD}{AD} = \frac{\text{area of }\Delta BCD}{\text{area of }\Delta BAD} = \frac{152+76}{120+360} = \frac{19}{40}$ (Δs with equal height)

$\frac{CD}{AD} = \frac{CD}{CD+AD} = \frac{19}{19+40} = \frac{19}{59}$; $\frac{CE}{CB} = \frac{CE}{CE+EB} = \frac{1}{1+2} = \frac{1}{3} \neq \frac{19}{59}$

$\because$ $\frac{CD}{CA} \neq \frac{CE}{CB}$, $\therefore$ ΔABC is not similar to ΔDEC.

25. (a) EF = DE (given), $\angle$BEC = $\angle$CDB (given),

$\angle$EFC = $\angle$DFB (vert. opp. $\angle$s), $\therefore$ ΔEFC $\cong$ ΔDFB (ASA).

(b) $\because$ AD = BD and AE = EC (given)

$\therefore$ DE // BC (mid-pt theorem.), $\therefore$ $\angle$DEF = $\angle$CBF (alt. $\angle$s, DE // BC),

$\angle$EDF = $\angle$BCF (alt. $\angle$s, DE // BC), $\angle$DFE = $\angle$CFB (vert. opp. $\angle$s),

$\therefore$ ΔDFE ~ ΔCFB (AAA). $\therefore$ The claim is agreed.

(c) (i) $DE = \frac{1}{2}BC$ (mid.-pt. thm) and $\Delta DFE \sim \Delta CFB$

$\therefore \frac{BC}{DE} = \frac{BF}{FE} = \frac{CF}{DF} = \frac{2}{1}$ (corr. sides, ~Δs)

$\frac{\text{Area of } \Delta CFB}{\text{Area of } \Delta DFE} = \left(\frac{2}{1}\right)^2$, $\therefore$ Area of ΔCFB = 4(25) = 100.

(ii) $\frac{\text{Area of } \Delta EFC}{\text{Area of } \Delta DFE} = \frac{FC}{DF} = \frac{2}{1}$ (Δs with equal height) ,

Area of ΔEFC = 2(25) = 50

(iii) Area of ΔDFB = Area of ΔEFC = 50 (≅Δs) ,

$\therefore$ Area of BCED = 100 + 50 + 50 + 25 = 225

(iv) DE // BC , $\therefore \angle AED = \angle ACB$ and $\angle ADE = \angle ABC$ (corr. ∠s, // lines) ,

$\therefore \Delta ADE \sim ABC$ (AAA), $\frac{BC}{DE} = \frac{2}{1}$, $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ADE} = \left(\frac{2}{1}\right)^2 = \frac{4}{1}$,

$\frac{\text{Area of } \Delta ABC}{\text{Area of BCDE}} = \frac{4}{4-1} = \frac{4}{3}$, $\therefore$ area of $\Delta ABC = \frac{4}{3}(255) = 300$.

26. Let *h* cm be the depth of the water.

$$\left(\frac{h}{12}\right)^3 = \frac{6\left[\frac{4}{3}\pi\left(\frac{2}{2}\right)^3\right]}{\frac{1}{3}(4)^2(12)} = \frac{8\pi}{64\pi} = \frac{1}{8}, \quad \therefore \frac{h}{12} = \frac{1}{2}, \quad h = 6.$$

Ans. The depth of the water is 6 cm.

27. (a) $\frac{\text{Volume of A}}{\text{Volume of B}} = \left(\sqrt{\frac{36}{25}}\right)^3 = \left(\frac{6}{5}\right)^3 = \frac{216}{125}$

Volume of A = $\frac{216}{216+125} \times 6\ 138\pi = \frac{216}{341} \times 6\ 138\pi = 3\ 888\pi$ cm^3

(b) Volume of B = $6\ 138\pi - 3\ 888\pi = 2\ 250\pi$ cm^3

Let *r* cm be the radius of B.

$\frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = 2\ 550\pi$, $\frac{2}{3}\pi r^3 = 2\ 550\pi$, $r^3 = 3\ 375$, $r = 15$.

Ans. The radius of B is 15 cm.

(c) Surface area of B = $\frac{1}{2}\left[4\pi(15)^2\right] + \pi(15)^2 = 450\pi + 225\pi = 675\pi$ cm^2.

$\therefore$ Total surface area of A and that of B = $\frac{675\pi}{25} \times (36+25) = 1\ 647\pi$ cm^2.

28. (a) Let L denote the large cone, S denote the small cone, and F denote the frustum.

$\text{Height}_L : \text{Heights}_S = 20 : (20 - 10) = 2 : 1$

$\therefore \text{Volume}_L : \text{Volumes}_S = 2^3 : 1^3 = 8 : 1$

$\therefore \text{Volume}_F : \text{Volume}_L = (8 - 1) : 8 = 7 : 8$

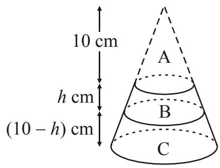
Volume of frustum = $\frac{1}{3}\pi(12)^2(20) \times \frac{7}{8} = 840\pi$ cm^3

(b) With the notation in the figure below:

$\text{Heigth}_A : \text{Height}_{A+B} : \text{Height}_{A+B+C} = 10 : 10 + h : 20$

$\text{Volume}_A : \text{Volume}_{A+B} : \text{Volume}_{A+B+C} = 10^3 : (10 + h)^3 : 20^3$

$= 1\ 000 : (10 + h)^3 : 8\ 000$



$\therefore$ Volume$_B$: Volume$_{B+C}$
$= [(10+h)^3 - 1\,000] : [8\,000 : 1\,000]$
$= [(10+h)^3 - 1\,000] : 7\,000$

$V = \text{Volume}_B = \dfrac{(10+h)^3 - 1\,000}{7\,000} \times 840\pi$

$= \left[\dfrac{840(10+h)^3}{7\,000} - 120\right]\pi = \left[\dfrac{3(10+h)^3}{25} - 120\right]\pi$

(c) $\left[\dfrac{3(10+h)^3}{25} - 120\right]\pi = 840\pi \times \dfrac{1}{2}$,

$\dfrac{3(10+h)^3}{25} = 540$, $(10+h)^3 = 4\,500$, $h = \sqrt[3]{4\,500} - 10$.

$\therefore$ The water level $= 10 - \left(\sqrt[3]{4\,500} - 10\right) = 3.49$ cm.

(d) (i) $h = 10 - 4 = 6$

Volume of water $= 840\pi - \left[\dfrac{3(10+h)^3}{25} - 120\right]\pi = 468.48\pi \text{ cm}^3$

(ii) When the frustum is upside down, h cm becomes the water depth.

$\left[\dfrac{3(10+h)^3}{25} - 120\right]\pi = 468.48\pi$, $(10+h)^3 = 4\,904$, $h = \sqrt[3]{4\,904} - 10 = 7.0$

Ans. The new water level is 7.0 cm.

Unit 11 Coordinate geometry: Distance & slope

1. (a) Slope $= \dfrac{0-(-30)}{24-0} = \dfrac{5}{4}$

(b) Slope $= \dfrac{25-(-75)}{-18-(-20)} = \dfrac{100}{2} = 50$

(c) Slope $= \dfrac{8-8}{2-(-3)} = \dfrac{0}{5} = 0$

(d) Slope $= \dfrac{10-4}{6-6} = \dfrac{6}{0}$ = undefined

(e) Slope $= \dfrac{-18.5-(-9.5)}{-8.5-0.5} = \dfrac{-9}{-9} = 1$

(f) Slope $= [3\frac{1}{3} - (-1\frac{1}{2})] \div [-1\frac{3}{4} - (-2\frac{1}{3})] = \dfrac{29}{6} \times \dfrac{12}{7} = \dfrac{58}{7}$

2. (a) Distance $= \sqrt{(5-2)^2 + (12-16)^2} = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

(b) Distance $= \sqrt{[-10-(-4)]^2 + [5-(-1)]^2} = \sqrt{(-6)^2 + 6^2} = \sqrt{72} = 6\sqrt{2}$

(c) Distance $= \sqrt{(-4\frac{2}{3} - 6\frac{4}{5})^2 + (\frac{3}{5} - \frac{1}{4})^2 +} = \sqrt{(-\frac{172}{15})^2 + (\frac{7}{20})^2} = 11.5$

(d) Distance $= \sqrt{(-4.2-1.7)^2 + [-10-(-6.8)]^2} = \sqrt{45.05} = 6.71$

3. (a) Slope of the line segment $= \dfrac{4-8}{9-3} = \dfrac{-4}{6} = \dfrac{-2}{3}$,

$\therefore$ slope of the lines parallel to it is $\dfrac{-2}{3}$.

(b) Slope of the line segment $= \dfrac{-2-5}{3-(-2)} = \dfrac{-7}{5}$,

$\therefore$ slope of the lines parallel to it is $-\frac{7}{5}$.

4. (a) Slope of the line segment = $\frac{1-(-1)}{7-6}=\frac{2}{1}=2$,

$\therefore$ slope of the lines perpendicular to it = $-1\div 2 = -\frac{1}{2}$

(b) the given line is vertical (x-coordinates equal),
$\therefore$ slope of the lines perpendicular to it = 0.

(c) the given line is horizontal (y-coordinates equal),
$\therefore$ slope of the lines perpendicular to it is undefined.

5. (a) $\frac{9-3}{-2-a}=\frac{3}{5}$, $30=-6-3a$, $3a=-36$, $\therefore\ a=-12$

(b) $(\frac{9-3}{-2-a})(-\frac{3}{2})=-1$, $18=-4-2a$, $2a=-22$, $\therefore\ a=-11$

6. (a) Slope = $\frac{-3-9}{10-(-10)}=\frac{-12}{20}=-\frac{3}{5}$

(b) Let PQ cuts the x-axis and y-axis at $(x, 0)$ and $(0, y)$ respectively.
$\frac{9-0}{-10-x}=-\frac{3}{5}$, $45=30+3x$, $3x=15$, $\therefore\ x=5$
$\frac{9-y}{-10-0}=-\frac{3}{5}$, $45-5y=30$, $5y=15$, $\therefore\ y=3$
Ans. PQ cuts the x-axis at (5, 0) and cuts the y-axis (0, 3).

7. $\sqrt{(8-k)^2+[-12-(-2)]^2}=k$,
$\sqrt{64-16k+k^2+(-10)^2}=k$, $k^2-16k+164=k^2$, $16k=164$, $\therefore\ k=10.25$

8. (a) the slope = $\frac{7-(-2)}{4-3}=9$, $\therefore$ the angle of inclination = $\tan^{-1}(9) = 83.7°$

(b) the slope = $\frac{1-(-8)}{-5-6}=\frac{9}{-11}$, $\tan^{-1}(\frac{9}{-11})=-39.3°$,
$\therefore$ the angle of inclination = 180° – 39.3° = 140.7°

9. $\frac{7-1}{-7-1}=\frac{t-1}{3-1}$, $\frac{6}{-8}=\frac{t-1}{2}$, $\frac{3}{-2}=t-1$, $t=-\frac{1}{2}$

10. $(\frac{9-18}{15-12})(\frac{16-12}{9-k})=-1$, $(\frac{-9}{3})(\frac{4}{9-k})=-1$, $12=9-k$, $\therefore k=-3$

11. (a) $m_{PQ}=\frac{1-(-2)}{5-1}=\frac{3}{4}$, $m_{QR}=\frac{0-(-2)}{4-1}=\frac{2}{3}$, $\because\ m_{PQ}\neq m_{QR}$, $\therefore$ P, Q, R are not collinear.

(b) $m_{AB}=\frac{3-(-3)}{5-2}=\frac{6}{3}=2$, $m_{BC}=\frac{3-1}{5-4}=2$, $\because\ m_{AB}=m_{BC}$, $\therefore$ A, B, C are collinear.

12. (a) BC = $\sqrt{(-2-8)^2+(3-1)^2}=\sqrt{104}$, AC = $\sqrt{[-2-(-3)]^2+[(3-(-2)]^2}=\sqrt{26}$,
AB = $\sqrt{[8-(-3)]^2+[1-(-2)]^2}=\sqrt{130}$, $\because$ BC^2+AC^2 = 104 + 26 = 130 = AB^2,

$\therefore$ ΔABC is right-angled.

(b) Area of $\Delta ABC = \frac{1}{2}(AC)(BC) = \frac{1}{2}(\sqrt{26})(\sqrt{104}) = \frac{1}{2}(\sqrt{26})(2\sqrt{26}) = 26$ sq. units

13. $m_{PQ} = \frac{-1-4}{3-1} = -\frac{5}{2}$, $m_{RS} = \frac{2-(-3)}{-4-(-2)} = -\frac{5}{2}$, $m_{QR} = \frac{-3-(-1)}{-2-3} = \frac{-2}{-5} = \frac{2}{5}$,

$m_{SP} = \frac{4-2}{1-(-4)} = \frac{2}{5}$. $\because$ $m_{PQ} \times m_{QR} = m_{RS} \times m_{QR} = m_{PQ} \times m_{SP} = m_{RS} \times m_{SP} = (\frac{-5}{2})(\frac{2}{5}) = -1$,

$\therefore$ PQ $\perp$ QR, RS $\perp$ QR, PQ $\perp$SP, PS $\perp$ SP. PQ $= \sqrt{(3-1)^2 + (-1-4)^2} = \sqrt{29}$,

QR $= \sqrt{[-3-(-1)]^2 + (-2-3)^2} = \sqrt{29}$, RS $= \sqrt{[2-(-3)]^2 + [-4-(-2)]^2} = \sqrt{29}$,

SP $= \sqrt{(4-2)^2 + [1-(-4)]^2} = \sqrt{29}$, $\therefore$ PQ = QR = RS = SP.

Ans. P, Q, R, S are the vertices of a square.

14. $\sqrt{(k-17)^2 + [3-(-29)]^2} = 40$, $(k-17)^2 + 32^2 = 40^2$, $(k-17)^2 = 576$,

$k - 17 = \pm\sqrt{576} = \pm 24$, $k = 24 + 17$ or $-24 + 17$, $\therefore$ $k = 41$ or -7.

15. PQ = 25 = $\sqrt{(x-6)^2 + (-12-8)^2}$, $25^2 = (x-6)^2 + 400$,

$(6-x)^2 = 225$, $\therefore$ $6 - x = 15$ or -15, $\therefore$ $x = 21$ or -9

16. Let coordinates of N be $(-6, n)$.

$n = \sqrt{[-6-(-2)]^2 + (n-8)^2}$, $n^2 = 16 + n^2 - 16n + 64$, $16n = 80$, $\therefore$ $n = 5$.

Ans. The coordinates of N are (–6, 5).

17. $\sqrt{[(n+1)-n]^2 + [2-(1-n)]^2} = \sqrt{26}$, $1 + (n+1)^2 = 26$, $(n+1)^2 = 25$,

$n+1 = \pm 5$, $n + 1 = 5$ or $n + 1 = -5$, $\therefore$ $n = 4$ or $n = -6$.

18. Let the coordinates of P be $(0, p)$. $\sqrt{(0-9)^2 + (p-3)^2} = \sqrt{[0-(-7)]^2 + [p-(-5)]^2}$,

$81 + p^2 - 6p + 9 = 49 + p^2 + 10p + 25$, $16p = 16$, $\therefore$ $p = 1$

Ans. The coordinates of P are (0, 1).

19. Distance = $\sqrt{(1-m^2)^2 + (2m-0)^2} = \sqrt{1 - 2m^2 + m^4 + 4m^2} = \sqrt{m^4 + 2m^2 + 1} = \sqrt{(m^2+1)^2}$

$= (m^2 + 1)$ units.

20. Let the coordinates of P be $(p, 0)$.

$p = \sqrt{(20-p)^2 + (-12-0)^2}$, $p^2 = 400 - 40p + p^2 + 144$, $40p = 544$, $p = 13.6$.

$\therefore$ Area of $\Delta OPQ = \frac{1}{2}(13.6)(12) = 81.6$ sq. units

21. AB = $\sqrt{[0-(-6)]^2 + (-2-0)^2} = \sqrt{40} = 2\sqrt{10}$, BC = AB = $2\sqrt{10}$,

$\therefore$ area of $\Delta ABC = \frac{1}{2}(2\sqrt{10})(6) = 6\sqrt{10}$ sq. units

22. AB = $\sqrt{[19-(-21)]^2 + (12-3)^2} = \sqrt{1681} = 41$

$\because$ AP : BP = 5 : 2, $\therefore$ AB : BP = 3 : 2, $\therefore$ BP = $\frac{2}{3}$AB = $\frac{2 \times 41}{3} = 27\frac{1}{3}$

23. Angle of inclination of L = 180°– 45° = 135°, $\therefore$ slope of L = tan 135° = –1

$\therefore$ $\frac{y-0}{x-(-3)} = -1$, $y = -x - 3$, $x = -3 - y$

24. (a) $m_{AD} = \dfrac{2-(-6)}{3-(-5)} = \dfrac{8}{8} = 1$, $\therefore$ angle of inclination of AD = $\tan^{-1}(1) = 45°$,

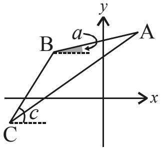
$\theta + 45° = 90°$ (ext. ∠ of Δ), $\theta = 45°$

(b) $m_{L_2} = \tan\beta = \dfrac{7-0}{4-2} = \dfrac{7}{2} = 3.5$, $\therefore$ $\beta \approx 74°$.

$m_{L_1} = \tan\alpha = \dfrac{4-0}{5-(-4)} = \dfrac{4}{9}$, $\therefore$ $\alpha \approx 24°$.

Let k be the vert . opp. ∠ of θ.
$k = \beta - \alpha = 74° - 24° = 50°$ (ext. ∠ of Δ); $\therefore$ $\theta = k = 50°$

25. Slope of AB = $\tan a = \dfrac{11-15}{-4-2} = \dfrac{-4}{-6} = \dfrac{2}{3}$, $\therefore$ $a = 33.69°$



slope of BC = $\tan c = \dfrac{-1-11}{-12-(-4)} = \dfrac{-12}{-8} = \dfrac{3}{2}$, $\therefore$ $c = 56.31°$

$\angle ABC = (180° - c) + a = 180° - 56.31° + 33.69° \approx 157°$

26. (a) When $x = 0$, $2(0) + 4y - 12 = 0$, $4y = 12, y = 3$.
When $y = 0$, $2x + 4(0) - 12 = 0$, $2x = 12$, $x = 6$.
Ans. The x-intercept is 6, y-intercept is 3.

(b) The line passes through (0, 3) and (6, 0), $\therefore$ slope = $\dfrac{3-0}{0-6} = \dfrac{3}{-6} = -\dfrac{1}{2}$

27. (a) AB = $\sqrt{(-3-0)^2+(6-4)^2} = \sqrt{(-3)^2+2^2} = \sqrt{13}$
BC = $\sqrt{(6-0)^2+(0-4)^2} = \sqrt{6^2+(-4)^2} = \sqrt{52} = 2\sqrt{13}$
AC = $\sqrt{[6-(-3)]^2+(0-6)^2} = \sqrt{9^2+(-6)^2} = \sqrt{117} = 3\sqrt{13}$

(b) AB + BC = $\sqrt{13}+2\sqrt{13} = 3\sqrt{13}$ = AC. $\therefore$ A, B C are collinear.

28. $\sqrt{(m-4)^2+(2-7)^2} = \sqrt{(n-4)^2+(2-7)^2}$,
$m^2 - 8m + 16 = n^2 - 8n + 16$, $m^2 - n^2 - 8m + 8n = 0$, $(m-n)(m+n) - 8(m-n) = 0$,
$(m-n)(m+n-8) = 0$, $\therefore$ $m + n - 8 = 0$ ($\because$ $m \neq n$), $m + n = 8$.

29. (a) AB $= \sqrt{[3-(-3)]^2+[2-(-6)]^2} = \sqrt{6^2+8^2} = \sqrt{100} = 10$
BC $= \sqrt{(5-3)^2+(-2-2)^2} = \sqrt{2^2+(-4)^2} = \sqrt{20} = 2\sqrt{5}$
AC $= \sqrt{[5-(-3)]^2+[-2-(-6)]^2} = \sqrt{8^2+4^2} = \sqrt{80} = 4\sqrt{5}$

slope of AB $= \dfrac{2-(-6)}{3-(-3)} = \dfrac{8}{6} = \dfrac{4}{3}$; slope of BC $= \dfrac{-2-2}{5-3} = \dfrac{-4}{2} = -2$;

slope of AC $= \dfrac{-2-(-6)}{5-(-3)} = \dfrac{4}{8} = \dfrac{1}{2}$.

(b) slope of BC × slope of AC $= -2 \times \dfrac{1}{2} = -1$, $\therefore$ BC ⊥ AC

Area of ΔABC $= \dfrac{1}{2}(2\sqrt{5})(4\sqrt{5}) = 20$ sq. units.

(c) $\sin\angle A = \dfrac{BC}{AB} = \dfrac{2\sqrt{5}}{10}$, $\therefore$ ∠A = 26.6°, ∠B = 90° − 26.6° = 63.4°

(d) Slope of the altitude × slope of AB = –1, $\therefore\ \frac{y-(-2)}{-7-5}\times\frac{4}{3}=-1$, $4(y+2)=36$, $y=7$

30. (a) $m_{AM}=\frac{(m-4)-2}{m-(-1)}==\frac{m-6}{m+1}$, $m_{BC}=\frac{1-(-5)}{6-(-2)}=\frac{3}{4}$

$\because$ AM ⊥ BC, $\therefore\ \frac{m-6}{m+1}\times\frac{3}{4}=-1$,

$3m-18=-4m-4$, $7m=14$, $\therefore\ m=2$, $m-4=-2$.

Ans. Coordinates of M are (2, –2).

(b) BC = $\sqrt{[6-(-2)]^2+[1-(-5)]^2}=\sqrt{100}=10$, AM = $\sqrt{(-2-2)^2+[2-(-1)]^2}=\sqrt{25}=5$

$\therefore$ Area of the ΔABC = $\frac{1}{2}$(BC)(AM) = $\frac{1}{2}$(10)(5) = 25 sq. units

(c) $m_{AB}=\frac{2-(-5)}{-1-(2)}=\frac{7}{1}$, AB = $\sqrt{[2-(-5)]^2+[-1-(-2)]^2}=\sqrt{50}=5\sqrt{2}$

Slope of CN = $-1\div m_{AB}=-1\div 7=-\frac{1}{7}$.

Area of ΔABC = $\frac{1}{2}$(AB) (CN) = 25,

$\therefore\ \frac{1}{2}(5\sqrt{2})$ (CN) = 25, $\therefore$ CN = $\frac{25\times 2}{5\sqrt{2}}=\frac{10}{\sqrt{2}}=\frac{10\sqrt{2}}{2}=5\sqrt{2}$.

31. $m_{L_2}=\frac{3-0}{0-4}=-\frac{3}{4}$, $m_{L_1}=\frac{b-0}{a-0}=\frac{b}{a}$.

$\because$ $L_1\perp L_2$, $\therefore\ m_{L_1}\times m_{L_2}=-1$, $\frac{b}{a}\times(-\frac{3}{4})=-1$, $\therefore\ b=\frac{4}{3}a$ ….(i)

$m_{L_2}=\frac{b-3}{a-0}=-\frac{3}{4}$, $4b-12=-3a$….(ii)

Sub. (i) into (ii),

$4(\frac{4}{3}a)-12=-3a$, $16a-36=-9a$, $25a=36$, $\therefore\ a=\frac{36}{25}$, $b=\frac{4}{3}(\frac{36}{25})=\frac{48}{25}$.

Ans. Coordinates of P are $(\frac{36}{25},\frac{48}{25})$.

32. $m_{PQ}=\frac{-5-1}{-1-2}=2$, $m_{SR}=\frac{k-3}{h+3}$, $\because$ PQ // SR, $\therefore\ 2=\frac{k-3}{h+3}$, $k=2h+9$ …(i)

$m_{PR}=\frac{3-1}{-3-2}=-\frac{2}{5}$, $m_{SQ}=\frac{k-(-5)}{h-(-1)}=\frac{k+5}{h+1}$;

$\because$ PR ⊥ SQ, $\therefore\ (-\frac{2}{5})(\frac{k+5}{h+1})=-1$, $2k+10=5h+5$, $2k=5h-5$ …(ii)

Sub. (i) into (ii), $2(2h+9)=5h-5$, $\therefore\ h=23$, $k=2(23)+9=55$.

Ans. Coordinates of S are (23, 55).

33. (a) $\because$ AM // DC, $\therefore\ \frac{0-2}{-6-0}=\frac{t-(-3)}{-2-1}$, $\frac{1}{3}=\frac{t+3}{-3}$, $\therefore\ t=-4$

$\because$ tan∠BAO = slope of AB = $\frac{1}{3}$, $\therefore$ ∠BAO = 18.43°

tan ∠OAD = $\frac{\text{vertical distance between D, A}}{\text{horizontal distance between D, A}} = \frac{0-4}{-2-(-6)} = \frac{4}{4} = 1$,

$\therefore$ ∠OAD = 45°

(b) AB = $\sqrt{[(-6)-0]^2+(0-2)^2} = \sqrt{40} = 4\sqrt{10}$

CD = $\sqrt{[-2-1]^2+[-4-(-3)]^2} = \sqrt{10}$

AD = $\sqrt{[(-6)-(2)]^2+[0-(-4)]^2} = \sqrt{32} = 4\sqrt{2}$

Let h be the perpendicular distance from D to AB.

∠DAB =∠DAO+∠BAO = 45° + 18.43° = 63.43°

$\therefore \sin\angle DAB = \frac{h}{AD}$, $\sin 63.43^\circ = \frac{h}{4\sqrt{2}}$, $h = 4\sqrt{2}\times\sin 63.43^\circ = 5.06$

(c) Area of trapezium ABCD = $\frac{1}{2}$(AB+CD) × h = $\frac{1}{2}(2\sqrt{10}+\sqrt{10})\times 5.06$ = 24.0 sq. units.

34. (a) $m_{OR} = \frac{4-0}{-6-0} = -\frac{2}{3}$. $\because$ OR ⊥ PQ, $\therefore$ slope of PQ = $-1\div(-\frac{2}{3}) = \frac{3}{2}$

(b) Let the coordinates of P and Q be (p, 0) and (0, q) respectively.

$\because$ P,R,Q are collinear, $\therefore$ $m_{PR} = m_{RQ} = m_{PQ} =$

$\frac{4-0}{-6-p} = \frac{3}{2}$, $8 = -18 - 3p$, $3p = -26$, $\therefore$ $p = -\frac{26}{3}$

$\frac{q-4}{0-(-6)} = \frac{3}{2}$, $2q - 8 = 18$, $2q = 26$, $\therefore$ $q = 13$

$\therefore$ Area of ΔOPQ = $\frac{1}{2}$(OP) (OQ) = $\frac{1}{2}(\frac{26}{3})(13)$ = 56.3 sq. units.

35. (a) Let the coordinates of E be (x, y).

$m_{BC} = \frac{6-4}{2-10} = \frac{2}{-8} = -\frac{1}{4}$, $\because$ AM⊥BC, $\therefore$ $m_{AE}\times m_{BC} = -1$,

$\frac{y-1}{x-1} = -1\div(-\frac{1}{4}) = 4$, $\therefore$ $y - 1 = 4x - 4$, $y = 4x - 3$…(i)

$m_{AC} = \frac{4-1}{10-1} = \frac{1}{3}$; $\because$ BN ⊥ AC, $\therefore$ $m_{BE}\times m_{AC} = -1$,

$\frac{y-6}{x-2} = -1\div\frac{1}{3} = -3$, $y - 6 = 6 - 3x$, $y = 12 - 3x$ …(ii)

Sub (i) into (ii), $4x - 3 = 12 - 3x$, $\therefore$ $x = \frac{15}{7}$, $\therefore$ $y = 4(\frac{15}{7}) - 3 = \frac{39}{7}$

Ans. The coordinates of E are $(\frac{15}{7}, \frac{39}{7})$.

(b) $m_{BA} = \frac{6-1}{2-1} = \frac{5}{1} = 5$; $\therefore$ slope of the altitude through C = $-1\div 5 = -\frac{1}{5}$.

But $m_{EC} = (\frac{39}{7} - 4) \div (\frac{15}{7} - 10) = \frac{11}{-55} = -\frac{1}{5}$.

$\because$ m_{EC} = slope of the altitude through C, $\therefore$ the altitude from C to AB passes through E.

36. (a) Coordinates of A = $(-r, 0)$, coordinates of B = $(r, 0)$.

(b) slope of AP = $\frac{y-0}{x-(-r)} = \frac{y}{x+r}$, slope of PB = $\frac{y-0}{x-r} = \frac{y}{x-r}$.

$\therefore$ slope of AP × slope of PB = $\frac{y}{x+r} \cdot \frac{y}{x-r} = \frac{y^2}{x^2-r^2}$.

However, $OP^2 = r^2$, $\therefore$ $x^2 + y^2 = r^2$, $x^2 - r^2 = -y^2$,

$\therefore$ slope of AP × slope of PB = $\frac{y^2}{-y^2} = -1$. $\therefore$ AP $\perp$ PD, i.e. $\angle APB = 90°$.

37. (a) Slope of PQ = Slope of QR, $\frac{1-6}{p-0} = \frac{6-(-4)}{0-(-8)}$, $\frac{-5}{p} = \frac{5}{4}$, $p = -4$

(b) Slope of $L_2 \times \frac{5}{4} = -1$. Slope of $L_2 = -\frac{4}{5}$.

(c) Suppose L_2 cuts the y-axis at $(0, c)$. $\frac{c-1}{0-(-4)} = -\frac{4}{5}$, $c = -\frac{4}{5}(4) + 1 = -\frac{11}{5}$

$\therefore$ The y-intercept of L_2 is $-\frac{11}{5}$

(d) The required area = $\frac{1}{2}\left[6 - \left(-\frac{11}{5}\right)\right](4) = \frac{82}{5}$

38. (a) Let $(s, 0)$ be the coordinates of S.

Slope of QS = Slope of PQ

$\frac{0-5}{s-24} = \frac{11-5}{36-24}$, $\frac{-5}{s-24} = \frac{1}{2}$, $s = 14$. $\therefore$ The coordinates of S is (14, 0)

(b) Slope of RS = $\frac{0-(-32)}{14-30} = -2$

$\therefore$ Slope of RS × Slope of PS = $(-2)\,(\frac{1}{2}) = -1$, $\therefore$ RS $\perp$ PS.

(c) (i) RS = $\sqrt{(30-14)^2 + (-32-0)^2} = \sqrt{16^2 + 32^2} = 16\sqrt{5}$

(ii) PS = $\sqrt{(36-14)^2 + (11-0)^2} = \sqrt{22^2 + 11^2} = 11\sqrt{5}$

Area of $\Delta PRS = \frac{1}{2}(RS)\,(PS) = \frac{1}{2}(16\sqrt{5})\,(11\sqrt{5}) = 440$

39. (a) Let $(e, 0)$ be the coordinates of E.

Slope of AE = Slope of AD.

$\frac{0-(-8)}{e-(-2)} = \frac{16-(-8)}{16-(-2)}$, $\frac{8}{e+2} = \frac{4}{3}$, $e = 4$.

$\therefore$ The coordinates of E are (4, 0)

(b) Slope of BE = Slope of CD.

$\dfrac{2-0}{b-4}=\dfrac{18-16}{-14-16}=-\dfrac{1}{15}$, $b=-26$.

(c) $\because$ Slope of BC = $\dfrac{18-2}{(-14)-(-26)}=\dfrac{4}{3}$ = Slope of AD

$\therefore$ BC // AD , i.e. BC // DE , and BE // CD, $\therefore$ BCDE is a parallelogram.

(d) BC = $\sqrt{[(-26)-(-14)]^2+(2-18)^2}=20$, AB = $\sqrt{[(-2)-(-26)]^2+[(-8)-2]^2}=26$

CD = $\sqrt{[(-14)-16]^2+(18-16)^2}=\sqrt{904}$, AD = $\sqrt{[16-(-2)]^2+[16-(-8)]^2}=30$

Perimeter of ABCD = AB + BC + CD + AD = 26 + 20 + $\sqrt{904}$ + 30

= 76 + $\sqrt{904}$ = 106.0665928... > 100

$\therefore$ The claim is agreed.

40. (a) (i) y-coordinates of H = 2

(ii) Let coordinates of H be (h, 2). Slope of BC = $\dfrac{2-(-5)}{6-0}=\dfrac{7}{6}$,

Slope of AH × slope of BC = −1, $\left(\dfrac{2-14}{h-0}\right)\left(\dfrac{7}{6}\right)=-1$, $h=14$.

$\therefore$ The coordinates of H are (14, 2).

(b) Let (a, b) be the coordinates of K.

Slope of CK = Slope of AC, $\dfrac{b-2}{a-6}=\dfrac{14-2}{0-6}$, $\dfrac{b-2}{a-6}=-2$, $b=14-2a$ ...(1)

Slope of BK = Slope of BH, $\dfrac{b-(-5)}{a-0}=\dfrac{2-(-5)}{14-0}$, $\dfrac{b+5}{a}=\dfrac{1}{2}$, $2b+10=a$ ...(2)

Sub (1) into (2) , $2(14-2a)+10=a$, $38-4a=a$, $a=\dfrac{38}{5}$.

$b=14-\dfrac{2(38)}{5}=-\dfrac{6}{5}$. $\therefore$ Coordinates of K = $\left(\dfrac{38}{5},\ -\dfrac{6}{5}\right)$.

BK = $\sqrt{\left(\dfrac{38}{5}-0\right)^2+\left[\left(-\dfrac{6}{5}\right)-(-5)\right]^2}=\sqrt{\dfrac{38^2}{5^2}+\dfrac{19^2}{5^2}}=\dfrac{19\sqrt{5}}{5}$.

Unit 12 Coordinate geometry: Point of division

1. (a) P = $\left(\dfrac{-8\times2+2\times3}{2+3},\ \dfrac{5\times2+(-1)\times3}{2+3}\right)=\left(\dfrac{-10}{5},\ \dfrac{7}{5}\right)=\left(-2,\ \dfrac{7}{5}\right)$

(b) P = $\left(\dfrac{-3\times1+(-12)\times2}{1+2},\ \dfrac{4\times1+0\times2}{1+2}\right)=\left(\dfrac{-27}{3},\ \dfrac{4}{3}\right)=\left(-9,\ \dfrac{4}{3}\right)$

(c) P = $\left(\dfrac{2\times5+\left(-\dfrac{1}{4}\right)\times4}{5+4},\ \dfrac{\left(\dfrac{3}{5}\right)\times5+6\times4}{5+4}\right)=\left(\dfrac{9}{9},\ \dfrac{27}{9}\right)=(1, 3)$

(d) AP : PB = 1 : $\dfrac{3}{7}$ = 7 : 3,

$\therefore$ P = $(\frac{-7\times3+3\times7}{3+7}, \frac{-7\times3+(-2)\times7}{3+7}) = (\frac{0}{10}, \frac{-35}{10}) = (0, -3.5)$

2. (a) $M = (\frac{7+(-5)}{2}, \frac{-2+(-8)}{2}) = (\frac{2}{2}, \frac{-10}{2}) = (1, -5)$

   (b) $M = (\frac{-3+0}{2}, \frac{2.5+(-5.5)}{2}) = (\frac{-3}{2}, \frac{-3}{2})$

   (c) $M = (\frac{\frac{4}{3}+4}{2}, \frac{-\frac{13}{6}+\frac{2}{3}}{2}) = (\frac{16}{3}\times\frac{1}{2}, -\frac{9}{6}\times\frac{1}{2}) = (\frac{8}{3}, -\frac{3}{4})$

3. Let the coordinates of A be (x, y).

   $0 = \frac{x\times4+8\times3}{3+4}$, $0 = 4x+24$, $\therefore$ $x = -6$

   $1 = \frac{y\times4+11\times3}{3+4}$, $7 = 4y+33$, $-26 = 4y$, $\therefore$ $y = -6.5$

   Ans. The coordinates of A are (–6, –6.5).

4. Let the coordinates of N be (x, y). $\frac{x+5.5}{2} = 1$, $x + 5.5 = 2$, $x = -3.5$.

   $\frac{y+(-3)}{2} = \frac{1}{2}$, $y - 3 = 1$, $y = 4$. *Ans. The coordinates of N are (–3.5, 4).*

5. (a) PQ : QR = (3 – 1) : (9 – 3) = 2 : 6 = 1 : 3

   (b) $k = \frac{7\times3+(-9)\times1}{3+1} = \frac{21-9}{4} = \frac{12}{4} = 3$

6. Let the coordinates of the points be (x_1, y_1) and (x_2, y_2).

   The ratio = 2 : 1, $\therefore$ $x_1 = \frac{-2\times2+7\times1}{2+1} = \frac{3}{3} = 1$, $y_1 = \frac{8\times2+(-2)\times1}{2+1} = \frac{14}{3}$.

   The ratio = 1 : 2, $\therefore$ $x_2 = \frac{-2\times1+7\times2}{2+1} = \frac{12}{3} = 4$, $y_2 = \frac{8\times1+(-2)\times2}{2+1} = \frac{4}{3}$.

   Ans. The coordinates of the points are (1, $\frac{14}{3}$) and (4, $\frac{4}{3}$).

7. (a) Let the ratio be $r : s$. The y-coordinate of the point of division = 0,

   $\therefore$ $s(-\frac{7}{3})+r(\frac{28}{5}) = 0$, $\frac{28}{5}r = \frac{7}{3}s$, $\frac{r}{s} = \frac{7}{3}\times\frac{5}{28} = \frac{5}{12}$. *Ans. The ratio is 5 : 12.*

   (b) The x-coordinate of the point of division = 0,

   $\therefore$ The ratio = $(5\frac{5}{6}-0)\div[0-(-1\frac{1}{4})] = \frac{35}{6}\div\frac{5}{4} = \frac{35}{6}\times\frac{4}{5} = \frac{14}{3}$

8. $\because$ AB : AC = 3 : 5, $\therefore$ AB : BC = 3 : 2.

   Let the coordinates of C be (x, y). $\frac{x\times3+8\times2}{3+2} = -1$, $3x+16 = -5$, $x = -7$.

   $\frac{y\times3+(-4)\times2}{3+2} = -7$, $3y-8 = -35$, $3y = -27$, $y = -9$.

   Ans. The coordinates of C are (–7, –9).

9. (a) The mid-point of PR = $(\frac{6+3}{2},\frac{9+(-12)}{2})$ = $(4.5,-1.5)$

(b) Let the coordinates of S be (x, y). $\because$ diagonals bisect each other,

$\therefore\ 4.5=\frac{12+x}{2},\quad x=-3;\quad -1.5=\frac{-30+y}{2},\quad y=-3+30=27.$

Ans. *The coordinates of S are (–3, 27).*

10. $\because\ \Delta PHK \sim \Delta PQR$ (AAA), $\therefore\ \frac{PH}{PQ}=\frac{HK}{QR}=\frac{1}{4}$,

PH : PQ = 1 : 4, PH : HQ = 1 : 3 $\therefore$ QH : HP = 3 : 1.

Let the coordinates of H be (x, y), $x=\frac{-7\times3+9}{3+1}=-3,\quad y=\frac{2\times3+12}{3+1}=4.5.$

Ans. *The coordinates of H are (–3, 4.5).*

11. Sub. $x = 0$ and $y = 0$ into the equation, we have $4y = 12$, $y = 3$ and $x = 12$,

$\therefore$ coordinates of A and B are (0, 3) and (12, 0) respectively.

Let the coordinates of P be (x, y). $x=\frac{0\times3+12\times1}{3+1}=3,\quad y=\frac{3\times3+0\times1}{3+1}=2.25.$

Ans. *The coordinates of P are (3, 2.25).*

12. $\because$ AC : CB = 3 : 2, $\therefore$ AB : BC = (3 – 2) : 2 = 1 : 2.

Let the coordinates of C be (x, y). $2=\frac{-2\times2+x\times1}{2+1},\quad 6=-4+x,\quad x=10.$

$5=\frac{3\times2+y\times1}{2+1},\quad 15=6+y,\quad y=9.$ *Ans.* *The coordinates of C are (10, 9).*

13. (a) Let the coordinates of P and Q be (x_P, y_P) and (x_Q, y_Q) respectively.

$x_P=\frac{7\times1+(-2)\times3}{1+3}=\frac{7-6}{4}=0.25,\quad y_P=\frac{6\times1+0\times3}{1+3}=\frac{6}{4}=1.5.$

$x_Q=\frac{7\times1+10\times3}{1+3}=\frac{37}{4}=9.25,\quad y_Q=\frac{6\times1+(-6)\times3}{1+3}=\frac{-12}{4}=-3.$

Ans. *The coordinates of P and Q are (0.25, 1.5) and (9.25, –3) respectively.*

(b) Slope of AB $=\frac{0-(-6)}{-2-10}=\frac{6}{-12}=-\frac{1}{2}$; slope of PQ $=\frac{-3-1.5}{9.25-0.25}=\frac{-4.5}{9}=-\frac{1}{2}$,

$\because$ slope of PQ = slope of AB, $\therefore$ PQ // AB

14. Let $P(x, y)$ be the intersection point of the diagonals.

$\because$ Diagonals bisect each other, $\therefore\ x=\frac{-3+5}{2}=1,\quad y=\frac{2+(-4)}{2}=-1.$

$\because$ P is a point on BD, $\therefore$ slope of BD = slope of BP $=\frac{7-(-1)}{8-1}=\frac{8}{7}$

15. $a-b=\frac{2\times4+5\times3}{4+3}-\frac{-9\times4+5\times3}{4+3}=\frac{8+15}{7}-\frac{-36+15}{7}=\frac{44}{7}$

16. $m=\frac{a\times a+(-b)\times b}{a+b}=\frac{a^2-b^2}{a+b}=\frac{(a+b)(a-b)}{a+b}=a-b$

$n = \dfrac{(-a)\times a + b\times b}{a+b} = \dfrac{b^2 - a^2}{a+b} = \dfrac{(b+a)(b-a)}{a+b} = b - a$

17. Let the coordinates of P and Q be $(0, y)$ and $(x, 0)$ respectively.

$a = \dfrac{0+x}{2}$, $\therefore\ x = 2a$; $b = \dfrac{y+0}{2}$, $\therefore\ y = 2b$

$\therefore$ PQ $= \sqrt{x^2+y^2} = \sqrt{(2a)^2+(2b)^2} = \sqrt{4a^2+4b^2} = 2\sqrt{a^2+b^2}$

18. Let AB $= x$. $\therefore$ CD $= 3x$; AD $= 5x$; BC $= 5x - x - 3x = x$; AC $= x + x = 2x$

$\therefore$ AB : BC = 1 : 1, AC : CD = 2 : 3.

Coordinates of B $= (\dfrac{-2+8}{1+1}, \dfrac{-3+7}{1+1}) = (\dfrac{6}{2}, \dfrac{4}{2}) = (3, 2)$.

Let the coordinates of D be (x_2, y_2).

$\dfrac{x_2\times 2 + (-2)\times 3}{2+3} = 8$, $2x_2 = 46$, $x_2 = 23$

$\dfrac{y_2\times 2 + (-3)\times 3}{2+3} = 7$, $\dfrac{2x_2 - 9}{5} = 7$, $2y_2 = 44$, $y_2 = 22$

Ans. The coordinates of B and D are (3, 2) and (23, 22) respectively.

19. Coordinates of P $= (\dfrac{1\times 2 + 6\times 1}{2+1}, \dfrac{2\times 2 + (-10)\times 1}{2+1}) = (\dfrac{8}{3}, \dfrac{-6}{3}) = (\dfrac{8}{3}, -2)$.

Sub. the coordinates into $3x - y + k = 0$, we have

$3(\dfrac{8}{3}) - (-2) + k = 0$, $8 + 2 + k = 0$, $\therefore\ k = -10$

20. Coordinates of the mid-point $= (\dfrac{3a - 1 + (-3)}{2}, \dfrac{4 + 5 - a}{2}) = (\dfrac{3a-4}{2}, \dfrac{9-a}{2})$.

Sub. the coordinates into the equation $x - 2y + 6 = 0$, we have

$(\dfrac{3a-4}{2}) - 2(\dfrac{9-a}{2}) + 6 = 0$, $3a - 4 - 2(9-a) + 12 = 0$, $5a - 10 = 0$, $\therefore\ a = 2$

21. (a) PC = CR and PA = AQ, $\therefore$ AC // QR // BR and AC $= \dfrac{1}{2}$QR = BR (mid-pt. thm.),

$\therefore$ ABRC is a parallelogram (opp. sides equal and //).

(b) Let the coordinates of R be (x, y).

$\because$ mid-point of BC = mid-point of AR, $\therefore\ \dfrac{x+(-4)}{2} = \dfrac{1+(-1)}{2}$, $x = 4$;

$\dfrac{y+6}{2} = \dfrac{7+3}{2}$, $y = 4$. *Ans. The coordinates of R are (4, 4).*

22. $-2\dfrac{3}{5} = \dfrac{(n-9)\times 3 + (m+4)\times 2}{3+2}$, $\dfrac{-13}{5} = \dfrac{3n - 27 + 2m + 8}{5}$, $6 = 3n + 2m$ …(i)

$-6\dfrac{4}{5} = \dfrac{(5m+1)\times 3 + n\times 2}{3+2}$, $-\dfrac{34}{5} = \dfrac{15m + 3 + 2n}{5}$, $-37 = 15m + 2n$ …(ii)

(i)×2 – (ii)×3, $\therefore\ 12 - (-111) = 4m - 45m$, $41m = -123$, $m = -3$

Sub $m = -3$ into (i), $6 = 3n + 2(-3)$, $12 = 3n$, $n = 4$.

Ans. *Coordinates of A = (4 – 9, –3 × 5+1) = (–5, –14);*

Coordinates of B are (–3 + 4, 4) = (1, 4).

23. Consider $3x + 5y - 30 = 0$:

when $x = 0$, $5y = 30$, $y = 6$; when $y = 0$, $3x = 30$, $x = 10$

Coordinates of P and Q are (0, 6) and (10, 0) respectively.

∵ ΔOPR and ΔORQ have the same heights, ∴ ratio of areas = PR : RQ = 1: 3

∴ Coordinates of R = $(\frac{0\times3+10\times1}{3+1}, \frac{6\times3+0\times1}{3+1}) = (\frac{10}{4}, \frac{18}{4}) = (2.5, 4.5)$

24. (a) Area of ΔOPB = $\frac{1}{3}$ × area of ΔOAB, $\frac{1}{2}nr = \frac{1}{3}\times\frac{1}{2}mn$, ∴ $r = \frac{m}{3}$

(b) Area of ΔOPA = $\frac{1}{3}$ × area of ΔOAB, $\frac{1}{2}ms = \frac{1}{3}\times\frac{1}{2}mn$, ∴ $s = \frac{n}{3}$

(c) BP : PQ = $(n - s) : (s - 0) = (n - \frac{n}{3}) : (\frac{n}{3} - 0) = \frac{2n}{3} : \frac{n}{3} = 2 : 1$

(d) $r = \frac{9\times2+0\times1}{2+1} = \frac{18}{3} = 6$, $m = 3(6) = 18$. *Ans. The coordinates of A are (18, 0).*

25. (a) Coordinates of M = $\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$.

Coordinates of N = $\left(\frac{c+0}{2}, 0\right) = \left(\frac{c}{2}, 0\right)$

(b) Coordinates of G = $\left(\frac{1(a)+2(\frac{c}{2})}{1+2}, \frac{1(b)+2(0)}{1+2}\right) = \left(\frac{a+c}{3}, \frac{b}{3}\right)$

(c) Coordinates of H = $\left(\frac{1(c)+2(\frac{a}{2})}{1+2}, \frac{1(0)+2(\frac{b}{2})}{1+2}\right) = \left(\frac{a+c}{3}, \frac{b}{3}\right)$

(d) Coordinates of R = $\left(\frac{a+c}{2}, \frac{b+0}{2}\right) = \left(\frac{a+c}{2}, \frac{b}{2}\right)$

∴ Coordinates of K = $\left(\frac{1(0)+2(\frac{a+c}{2})}{1+2}, \frac{1(0)+2(\frac{b}{2})}{1+2}\right) = \left(\frac{a+c}{3}, \frac{b}{3}\right)$

(e) PN, QM and OR are the medians of ΔOPQ. ∵ Coordinates of G, H and K are the same,
∴ the medians are concurrent, and their point of intersection, centroid, divides each of them in the ratio 2 : 1.

26. (a) AB = OC = c (prop. of rhombus), ∴ Coordinates of B = $(a + c, b)$

(b) mid-point of AC = $(\frac{a+c}{2}, \frac{b+0}{2}) = (\frac{a+c}{2}, \frac{b}{2})$

mid-point of OB = $(\frac{0+(a+c)}{2}, \frac{0+b}{2}) = (\frac{a+c}{2}, \frac{b}{2})$

∵ mid-point of AC = mid-point of OB, ∴ AC and OB bisect each other,
i.e. the diagonals of a rhombus bisect each other.

(c) Slope of OB = $\dfrac{b-0}{(a+c)-0}=\dfrac{b}{a+c}$, slope of AC = $\dfrac{b-c}{a-s}=\dfrac{b}{a-c}$

$OA^2 = OC^2$, $\therefore\ (a-0)^2+(b-0)^2=c^2$, $\therefore\ a^2+b^2=c^2$, $a^2-c^2=-b^2$

slope of OB × slope of AC = $\dfrac{b}{a+c}\times\dfrac{b}{a-c}=\dfrac{b^2}{a^2-c^2}=\dfrac{b^2}{-b^2}=-1$,

$\therefore$ OB ⊥ AC, i.e. diagonals of a rhombus are perpendicular to each other.

27. (a) (i) AD : DC = 9 : 4, $\therefore\ 3d=\dfrac{9(52)+4(0)}{9+4}$, $d = 12$.

(ii) Let A be $(a, 0)$.

$4(12)=\dfrac{9(96)+4(a)}{9+4}$, $a=\dfrac{-240}{4}=-60$

Ans. The coordinates of A are (–60, 0).

(b) (i) AE : ED = [0 – (–60)] : (4 × 12 – 0) = 5 : 4

(ii) Let E be $(0, c)$. $c=\dfrac{5(3\times 12)+4(0)}{5+4}=20$

Ans. The coordinates of E are (0, 20)

(iii) area of ΔAEB : area of ΔACB = $\dfrac{1}{2}$ (AB) (20) : $\dfrac{1}{2}$ (AB) (52)

= 20 : 52 = 5 : 13

(c) Let (p, q) be the coordinates of P.

PC : AC = area of ΔPBC : area of ΔABC = 2 : 1

$\therefore\ \dfrac{p\times 1+(-60)\times 2}{1+2}=96$, $p = 408$; $\dfrac{q\times 1+0\times 2}{1+2}=52$, $q = 156$

$\therefore$ The coordinates of P are (408, 156).

28. (a) Coordinates of R = $\left(\dfrac{0\times 1+21\times 2}{1+2},\dfrac{0\times 1+24\times 2}{1+2}\right)$ = (14, 16).

Coordinates of S = $\left(\dfrac{33+21}{2},\dfrac{0+24}{2}\right)$ = (27, 12).

(b) Coordinates of T = $\left(\dfrac{14r+33\times 1}{1+r},\dfrac{16r+0\times 1}{1+r}\right)=\left(\dfrac{14r+33}{1+r},\dfrac{16r}{1+r}\right)$

(c) Slope of OA = Slope of OS

$\dfrac{y-0}{x-0}=\dfrac{12-0}{27-0}$, $\dfrac{y}{x}=\dfrac{4}{9}$, $y=\dfrac{4}{9}x$ …(❖)

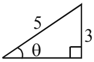
(d) Substitute $x=\dfrac{14r+33}{1+r}$ and $y=\dfrac{16r}{1+r}$ into (❖).

$\dfrac{16r}{1+r}=\dfrac{4}{9}\left(\dfrac{14r+33}{1+r}\right)$, $36r = 14r + 33$, $r=\dfrac{3}{2}$,

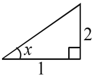
$\therefore$ RT : TP $=1:\dfrac{3}{2}=2:3$

Unit 13 Trigonometric relations

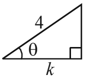
1. (a) $\sin\theta = 0.6 = \frac{3}{5}$, $\cos\theta = \frac{\sqrt{5^2 - 3^2}}{5} = \frac{4}{5} = 0.8$



(b) $\tan x = 2 = \frac{2}{1}$, $\sin x = \frac{2}{\sqrt{2^2 + 1^2}} = \frac{2}{\sqrt{5}}$



(c) $\cos\theta = \frac{k}{4}$, $\tan\theta = \frac{\sqrt{4^2 - k^2}}{k} = \frac{\sqrt{16 - k^2}}{k}$



2. $\frac{4\sin\theta - 3\cos\theta}{10\sin\theta + \cos\theta} = \frac{4\tan\theta - 3}{10\tan\theta + 1} = \frac{4(\frac{5}{13}) - 3}{10(\frac{5}{13}) + 1} = \frac{20 - 39}{50 + 13} = \frac{-19}{63}$

3. (a) $= (\frac{1}{2})(\frac{1}{2}) + (\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2}) = \frac{1}{4} + \frac{3}{4} = 1$

(b) $\frac{\sin^2 45°}{\cos 60°} - \frac{\tan^2 30°}{\cos 30°} = \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{\frac{1}{2}} - \frac{\left(\frac{1}{\sqrt{3}}\right)^2}{\frac{\sqrt{3}}{2}} = 1 - \frac{2}{\sqrt{3}} = 1 - \frac{2\sqrt{3}}{9} = \frac{9 - 2\sqrt{3}}{9}$

(c) $= (\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}})^2 = (\frac{2}{\sqrt{3}})^2 = \frac{4}{3}$

(d) $= 1 \div \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \sqrt{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

4. (a) $\tan\theta = \frac{\cos 47°}{\sin 47°} = \frac{1}{\tan 47°} = \tan(90° - 47°) = \tan 43°$, $\therefore\ \theta = 43°$

(b) $\tan^2\theta = \frac{1}{3}, \tan\theta = \frac{1}{\sqrt{3}} = \tan 30°$, $\therefore\ \theta = 43°$

(c) $\sin\theta = \cos(40° + \theta) = \sin[90° - (40° + \theta)] = \sin(50° - \theta)$,

$\therefore\ \theta = 50° - \theta$, $\theta = 25°$

5. (a) $= \sin\theta\cos\theta(\sin^2\theta + \cos^2\theta) = \sin\theta\cos\theta(1) = \sin\theta\cos\theta$

(b) $= \sin^2\theta(\frac{\sin^2\theta}{\cos^2\theta} + 1) = \sin^2\theta(\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}) = \sin^2\theta(\frac{1}{\cos^2\theta}) = \tan^2\theta$

(c) $= \frac{(\sin\theta - 1) - (1 + \sin\theta)}{(1 + \sin\theta)(\sin\theta - 1)} = \frac{-2}{-(1 - \sin^2\theta)} = \frac{2}{\cos^2\theta}$

(d) $= \frac{\sin\theta + \cos\theta}{1 - \frac{\sin^2\theta}{\cos^2\theta}} = \frac{\cos^2\theta(\sin\theta + \cos\theta)}{\cos^2\theta - \sin^2\theta} = \frac{\cos^2\theta(\sin\theta + \cos\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$

$= \frac{\cos^2\theta}{\cos\theta - \sin\theta} = \frac{\cos\theta}{1 - \tan\theta}$

6. (a) L.H.S. $= \cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$

$= (1 - \sin^2\theta - \sin^2\theta)(1) = 1 - 2\sin^2\theta$ = R.H.S.

(b) L.H.S. $= (1 - \cos y)(1 + \cos y) = 1 - \cos^2 y = \sin^2 y$

R.H.S. $= \cos^2 y \tan^2 y = \cos^2 y \frac{\sin^2 y}{\cos^2 y} = \sin^2 y$ = L.H.S.

(c) R.H.S. $= \dfrac{\tan\theta}{1+\tan^2\theta} = \dfrac{\frac{\sin\theta}{\cos\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}} = \dfrac{\sin\theta\cos\theta}{\cos^2\theta+\sin^2\theta} = \sin\theta\cos\theta$ = L.H.S.

7. (a) $\cos^2 35^\circ + \cos^2 55^\circ = \sin^2(90^\circ - 35^\circ) + \cos^2 55^\circ = \sin^2 55^\circ + \cos^2 55^\circ = 1$

(b) $\tan 59^\circ \times \tan 31^\circ = \dfrac{1}{\tan(90^\circ - 59^\circ)} \times \tan 31^\circ = \dfrac{1}{\tan 31^\circ} \times \tan 31^\circ = 1$

(c) $\sin^2 22^\circ - \cos^2 68^\circ = \cos^2(90^\circ - 22^\circ) - \cos^2 68^\circ = \cos^2 68^\circ - \cos^2 68^\circ = 0$

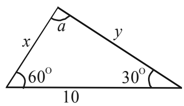
8. (a) $= \dfrac{\tan\theta - \frac{1}{\tan\theta}}{\tan\theta + \frac{1}{\tan\theta}} = \dfrac{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}} = \dfrac{\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}} = \dfrac{\sin^2\theta - \cos^2\theta}{\sin^2\theta + \cos^2\theta} = \sin^2\theta - \cos^2\theta$

(b) $= \dfrac{2\cos\theta\sin^2\theta}{\cos^2\theta\sin\theta} = \dfrac{2\sin\theta}{\cos\theta} = 2\tan\theta$

(c) $\tan(30^\circ - x) \times \tan(60^\circ + x)$

$= \dfrac{1}{\tan[90^\circ - (30^\circ - x)]} \times \tan(60^\circ + x) = \dfrac{1}{\tan(60^\circ + x)} \times \tan(60^\circ + x) = 1$

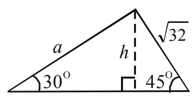
9. (a) $\because\ a = 180^\circ - 60^\circ - 30^\circ = 90^\circ$



$\therefore\ \sin 30^\circ = \dfrac{x}{10},\quad x = 10\sin 30^\circ = 10(\dfrac{1}{2}) = \underline{\underline{5}}$

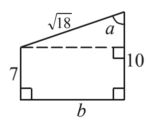
$\therefore\ \sin 60^\circ = \dfrac{y}{10},\quad y = 10\sin 60^\circ = 10(\dfrac{\sqrt{3}}{2}) = \underline{\underline{5\sqrt{3}}}$

(b) $\sin 45^\circ = \dfrac{h}{\sqrt{32}},\quad h = \sqrt{32}\sin 45^\circ,\quad \sin 30^\circ = \dfrac{h}{a},\quad a = \dfrac{h}{\sin 30^\circ},$



$\therefore\ a = \dfrac{\sqrt{32}\sin 45^\circ}{\sin 30^\circ} = \dfrac{(4\sqrt{2})(\frac{1}{\sqrt{2}})}{\frac{1}{2}} = 2\times 4 = \underline{\underline{8}}$

(c) $\because\ \cos a = \dfrac{10-7}{\sqrt{18}} = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}},\quad \therefore\ a = \underline{\underline{45^\circ}}$



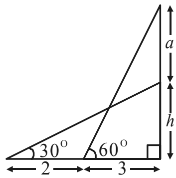
$\because\ \sin a = \dfrac{b}{\sqrt{18}},\quad b = \sqrt{18}\sin 45^\circ = 3\sqrt{2}(\dfrac{1}{\sqrt{2}}) = 3$

(d) $\dfrac{x+2}{12} = \cos 30^\circ,\quad x+2 = \dfrac{\sqrt{3}}{2}\cdot 12,\quad x = 6\sqrt{3} - 2$

10. In ΔCDB, $\mathrm{DC} = 9\tan 30^\circ = \dfrac{9\sqrt{3}}{3} = 3\sqrt{3}$

In ΔCAB, $x + \mathrm{DC} = 9\tan 60^\circ = 9\sqrt{3}$,

$x = 9\sqrt{3} - \mathrm{DC} = 9\sqrt{3} - 3\sqrt{3} = 6\sqrt{3}$



11. $\tan 30° = \frac{h}{2+3} = \frac{h}{5}$, $h = 5\tan 30° = \frac{5\sqrt{3}}{3}$

$\tan 60° = \frac{a+h}{3}$, $a + h = 3\sqrt{3}$, $a = 3\sqrt{3} - h = 3\sqrt{3} - \frac{5\sqrt{3}}{3} = \frac{4\sqrt{3}}{3}$

12. AD = $80 - \frac{18}{\tan 30°} - \frac{18}{\tan 45°} = 80 - 18\sqrt{3} - 18 = (62 - 18\sqrt{3})$ cm

13. $\because$ $\tan\theta = \frac{1}{\tan(90° - \theta)}$, $\therefore$ $\tan\theta \times \tan(90° - \theta) = 1$.

The given expression

= (tan 1° × tan 89°) × (tan 2° × tan 88°) × ...... × (tan 44° tan 46°) × tan 45°

= (1) × (1) × ...... × (1) tan 45° = 1

14. (a) L.H.S. $= \cos^4\theta - \sin^4\theta + 2\sin^2\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) + 2\sin^2\theta$

$= \cos^2\theta - \sin^2\theta + 2\sin^2\theta = \cos^2\theta + \sin^2\theta = 1$ = R.H.S.

(b) L.H.S. $= \frac{\cos\theta}{1+\sin\theta} \cdot \frac{1-\sin\theta}{1-\sin\theta} = \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} = \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} = \frac{1-\sin\theta}{\cos\theta}$ = R.H.S.

(c) L.H.S. $= (\sin^2\theta)^3 + (\cos^2\theta)^3$

$= (\sin^2\theta + \cos^2\theta)(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta)$

$= (1)(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta)$

$= (\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta) - 3\sin^2\theta\cos^2\theta$

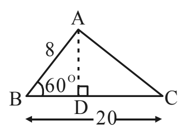
$= (\sin^2\theta + \cos^2\theta)^2 - 3\sin^2\theta\cos^2\theta = (1)^2 - 3\sin^2\theta\cos^2\theta$ = R.H.S.

15. (a) $= \frac{\cos^2\theta + (\sin\theta + 1)^2}{\cos\theta(1+\sin\theta)} = \frac{\cos^2\theta + \sin^2\theta + 2\sin\theta + 1}{\cos\theta(1+\sin\theta)}$

$= \frac{2 + 2\sin\theta}{\cos\theta(1+\sin\theta)} = \frac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)} = \frac{2}{\cos\theta}$

(b) $= \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta - 1}{\cos\theta}} = \frac{\sin^2\theta}{\cos\theta(\cos^2\theta - 1)} = \frac{\sin^2\theta}{\cos\theta(-\sin^2\theta)} = \frac{-1}{\cos\theta}$

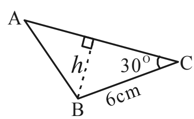
(c) $= \frac{1 + \cos\theta - (1 - \cos^2\theta)}{1+\cos\theta} = \frac{\cos\theta(1+\cos\theta)}{1+\cos\theta} = \cos\theta$



16. In ΔABD, $AD = AB\sin 60° = (8)(\frac{\sqrt{3}}{2}) = 4\sqrt{3}$,

$BD = AB\cos 60° = (8)(\frac{1}{2}) = 4$, $\therefore$ DC = BC – BD = 20 – 4 = 16,

$AC = \sqrt{AD^2 + DC^2} = \sqrt{(4\sqrt{3})^2 + (16)^2} = \sqrt{16(3+16)} = 4\sqrt{19}$



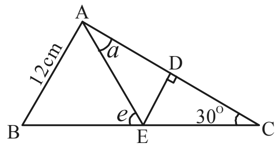
17. h = 6 sin 30° = 3 cm.

Area of $\Delta ABC = \frac{1}{2} \times h \times AC = 27$, $AC = \frac{2\times 27}{h} = \frac{54}{3} = 18\,cm$

18. ΔABE is equilateral, $\therefore$ e = 60°, AE = 12 cm

$a = e$ – 30° (ext. ∠ of Δ), a = 60 ° – 30° = 30°

$\therefore$ ΔAEC is isosceles (base ∠s equal), $\therefore$ AD = DC



In ΔAED, $ED = AE\sin a = 12\sin 60° = 12(\frac{\sqrt{3}}{2}) = 6\sqrt{3}$ cm

$AD = AE\cos a = 12\cos 60° = 12(\frac{1}{2}) = 6$ cm

Area of $\Delta AEC = 2\times(\frac{1}{2}\times AD\times ED) = 6\times 6\sqrt{3} = 36\sqrt{3}$ cm^2

19. $\because \sin(m+n) = \frac{\sqrt{3}}{2}$, $\therefore m+n = 60°$ …(1)

$\because \cos(m-n) = \frac{\sqrt{3}}{2}$, $\therefore m-n = 30°$ …(2),

(2) + (1), $2m = 90°$, $\therefore m = 45°$. From (1), $45° + n = 60°$, $\therefore n = \underline{\underline{15°}}$

20. $18\cos^2 x + 5(1-\cos^2 x) = 9$, $13\cos^2 x = 4$, $\cos x = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}}$, $\therefore \tan x = \frac{\sqrt{13-2^2}}{2} = \frac{3}{2}$

21. $a^2\sin^2\theta = 2^2 = 4$ and $a^2\cos^2\theta = 3^2 = 9$, $\therefore a^2\sin^2\theta + a^2\cos^2\theta = 4+9 = 13$,

$\therefore a^2(\sin^2\theta + \cos^2\theta) = 13$, $a^2(1) = 13$, $a = \sqrt{13}$

22. $\sin\theta - \sqrt{2}\cos\theta = 0$, $\sin\theta = \sqrt{2}\cos\theta$, $\tan\theta = \sqrt{2} = \frac{\sqrt{2}}{1}$, $\sin\theta\cos\theta = (\frac{\sqrt{2}}{\sqrt{3}})(\frac{1}{\sqrt{3}}) = \frac{\sqrt{2}}{3}$

23. $(\sin\theta - \cos\theta)^2 = (\frac{1}{3})^2$, $(\sin^2\theta + \cos^2\theta) - 2\sin\theta\cos\theta = \frac{1}{9}$,

$1 - 2\sin\theta\cos\theta = \frac{1}{9}$, $2\sin\theta\cos\theta = \frac{8}{9}$, $\therefore \sin\theta\cos\theta = \frac{4}{9}$

24. $(\sin\theta + \cos\theta)^2 = (\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta = 1 + 2(\frac{13}{10}) = \frac{18}{5}$

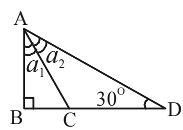
$\therefore \sin\theta + \cos\theta = \sqrt{\frac{18}{5}} = \frac{3\sqrt{2}}{\sqrt{5}} = \frac{3\sqrt{10}}{5}$

25. $$\frac{1}{\frac{1}{\cos\theta}-1} - \frac{1}{\frac{1}{\cos\theta}+1} = \frac{1}{\frac{1-\cos\theta}{\cos\theta}} - \frac{1}{\frac{1+\cos\theta}{\cos\theta}} = \cos\theta\left(\frac{1}{1-\cos\theta} - \frac{1}{1+\cos\theta}\right)$$

$$= \cos\theta\left[\frac{(1+\cos\theta)-(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}\right] = \cos\theta\left(\frac{2\cos\theta}{1-\cos^2\theta}\right) = \frac{2\cos^2\theta}{\sin^2\theta} = \frac{2}{\tan^2\theta} = \frac{2}{a^2}$$

26. $\tan 30° = \frac{BC}{AC} = \frac{BC}{2MC}$, $\therefore \frac{\sqrt{3}}{3} = \frac{1}{2}(\frac{BC}{MC})$, $\frac{2\sqrt{3}}{3} = \frac{BC}{MC}$

But $\tan\theta = \frac{BC}{MC}$, $\therefore \tan\theta = \frac{2\sqrt{3}}{3}$



27. $a_1 = a_2 = \frac{180° - 90° - 30°}{2} = 30°$, $\therefore$ AC = CD (sides opp. eq. ∠s)

But $\frac{BC}{AC} = \sin a_1 = \sin 30° = \frac{1}{2}$,

$\therefore 2BC = AC = CD$, $\frac{BC}{CD} = \frac{1}{2}$, BC : CD = 1 : 2

28. Let BD = a, $\therefore$ DC = $3a$; BC = $a + 3a = 4a$.

$\tan 60° = \dfrac{\text{DC}}{\text{AC}} = \dfrac{3a}{\text{AC}}$, AC = $\dfrac{3a}{\tan 60°} = \dfrac{3a}{\sqrt{3}} = \sqrt{3}a$. $\tan\theta = \dfrac{\text{AC}}{\text{BC}} = \dfrac{\sqrt{3}a}{4a} = \dfrac{\sqrt{3}}{4}$

29. AM ⊥ BM and ∠BAM = 45° (prop. of square), ∠BPM = 60° (equil. Δ).

In ΔBMP, BM = PM tan 60° = $\sqrt{3}$PM. In ΔBMA, BM = AM tan 45° = AM.

$\therefore$ AM = $\sqrt{3}$PM. $\because$ ΔABM ~ ΔPQM, $\therefore$ $\dfrac{\text{AB}}{\text{PQ}} = \dfrac{\text{AM}}{\text{PM}}$, $\dfrac{\text{AB}}{y} = \dfrac{\sqrt{3}\text{PM}}{\text{PM}}$, AB = $\sqrt{3}y$.

30. (a) ∠ANM = ∠ACB = 90°, $\therefore$ MN//BC (corr. ∠s equal),

$\therefore$ $\theta_1 = \theta_2$ (alt. ∠s, MN//BC)

(b) $\tan\angle A = \dfrac{\text{MN}}{\text{AN}} = 1$, $\therefore$ ∠A = 45°. In ΔABC, $\dfrac{\text{BC}}{\text{AC}} = \tan 45°$, $\therefore$ BC = AC

MN//BC (proved) and MB = AM (given)

$\therefore$ NC = AN (intercept theorem) $= \dfrac{1}{2}$AC. $\tan\theta_1 = \tan\theta_2 = \dfrac{\text{NC}}{\text{BC}} = \dfrac{\frac{1}{2}\text{AC}}{\text{AC}} = \dfrac{1}{2}$

31. (a) L.H.S. = $\dfrac{\sin\alpha\cos\alpha}{1+\cos\alpha} - \dfrac{\cos\alpha\sin\alpha}{1-\cos\alpha} = \sin\alpha\cos\alpha\left(\dfrac{1}{1+\cos\alpha} - \dfrac{1}{1-\cos\alpha}\right)$

$= \sin\alpha\cos\alpha\left[\dfrac{(1-\cos\alpha)-(1+\cos\alpha)}{(1+\cos\alpha)(1-\cos\alpha)}\right] = \sin\alpha\cos\alpha\left(\dfrac{-2\cos\alpha)}{(1-\cos^2\alpha)}\right)$

$= \dfrac{-2\sin\alpha\cos^2\alpha}{\sin^2\alpha} = -2\cos\alpha\left(\dfrac{\cos\alpha}{\sin\alpha}\right) = -2\cos\alpha\left(\dfrac{1}{\tan\alpha}\right)$

$= -2\cos\alpha\tan(90°-\alpha)$ = R.H.S.

(b) Let $\alpha = 90° - \beta$, $\therefore 90° - \alpha = \beta$.

L.H.S. = $\dfrac{\sin(90°-\alpha)\sin\alpha}{1+\cos\alpha} - \dfrac{\cos(90°-\alpha)\cos\alpha}{1-\cos\alpha} = -2\cos\alpha\tan(90°-\alpha)$ [from (a)]

$= -2\cos(90°-\beta)\tan\beta = -2\sin\beta\tan\beta$ = R.H.S.

32. (a) $(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + 2\sin\theta\cos\theta$

$\therefore$ A = 1, B = 2

(b) L.H.S. = $\dfrac{\cos^2\theta - \cos^2(90°-\theta)}{2\sin\theta\sin(90°-\theta)+1} = \dfrac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta+1}$

$= \dfrac{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{(\sin\theta+\cos\theta)^2}$ [from (a)]

$= \dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \dfrac{\frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}} = \dfrac{1-\frac{\sin\theta}{\cos\theta}}{1+\frac{\sin\theta}{\cos\theta}} = \dfrac{1-\tan\theta}{1+\tan\theta}$ = R.H.S.

33. (a) L.H.S. = $\sin^4 x + \cos^4 x$

$= (\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x\cos^2 x - 2\sin^2 x\cos^2 x$

$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$= 1 - 2\sin^2 x\cos^2 x$

= R.H.S.

(b) $$\text{L.H.S.} = \frac{1}{\sin^3(90° - y)\cos y} - 2\tan^2 y = \frac{1}{\cos^3 y \cos y} - 2\left(\frac{\sin^2 y}{\cos^2 y}\right)$$

$$= \frac{1}{\cos^4 y} - \frac{2\sin^2 y}{\cos^2 y} = \frac{1 - 2\sin^2 y\cos^2 y}{\cos^4 y} = \frac{\cos^4 y + \sin^4 y}{\cos^4 y} \quad \text{[by (a)]}$$

$$= 1 + \frac{\sin^4 y}{\cos^4 y} = 1 + \tan^4 y = 1 + \frac{1}{\tan^4(90° - y)} = \text{R.H.S.}$$

Unit 14 Applications of trigonometry

1. (a) area of ABCD = $14 \times 9\sin 60° = 109.1$ cm^2

   (b) area of ΔPQR = $\dfrac{20 \times 15\sin 72°}{2} = 142.7$ cm^2

   (c) the height $= \sqrt{26^2 - \left(\dfrac{18}{2}\right)^2} = \sqrt{595}$, $\quad\therefore$ area of ΔMNR = $\dfrac{18 \times \sqrt{595}}{2} = 219.5$ cm^2

   (d) area of ΔABC = $\dfrac{5 \times 5\sin 60°}{2} = 10.8$ cm^2

2. The hexagon can be cut into 6 congruent isosceles triangles whose vertical angle $= \dfrac{360°}{6} = 60°$.

   Let h cm be the height of the isosceles triangle.

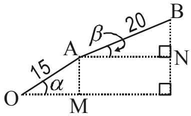
   $\tan\dfrac{60°}{2} = \dfrac{8 \div 2}{h}$, $\quad h = \dfrac{4}{\tan 30°}$. $\quad\therefore$ Area of the hexagon $= 6 \times \left(\dfrac{1}{2} \times 8 \times \dfrac{4}{\tan 30°}\right) = 96\sqrt{3}$ cm^2.

3. Let θ be the angle of inclination, $\quad \tan\theta = \dfrac{1}{15}$, $\quad \theta = 3.81°$.

   Ans. The angle of inclination is 3.81°.

4. Horizontal distance = $\sqrt{116^2 - 30^2} = \sqrt{12556}$, $\quad\therefore$ Gradient = $\dfrac{30}{\sqrt{12556}} = 0.268$.

5. $\tan\alpha = \dfrac{1}{10}$, $\alpha = 5.71°$; $\quad\therefore$ AM = 15 sin 5.71°;

   $\tan\beta = \dfrac{1}{12}$, $\beta = 4.76°$; $\quad\therefore$ BN = 20 sin 4.76°;

   ∴ Vertical distance = 15 sin 5.71° + 20 sin 4.76° = 3.15 km

6. Vertical distance = 400 − 350 = 50,
   horizontal distance = 25000 × 4 ÷ 100 = 1000.

   Let θ be the angle of inclination, $\tan\theta = \dfrac{50}{1000}$, $\quad \theta = 2.86°$.

   Ans. The angle of inclination is 2.86°.

7. Fiona's height = 2.14 tan 35°.

   $\dfrac{\text{Fiona's height}}{\text{the new shadow}} = \tan 60°$, $\quad$ the new shadow = $\dfrac{2.14\tan 35°}{\tan 60°} = 0.865$

   Ans. The length of the new shadow is 0.865m.

8. The angles of elevation are also 47° and 63°.

$\therefore$ Distance between A and B = $\dfrac{22}{\tan 47°} + \dfrac{22}{\tan 63°}$ = 31.7 m

9. The angles of elevation are also 24° and 36°.

$\therefore$ Distance between the cars = $\dfrac{120}{\tan 24°} - \dfrac{120}{\tan 36°}$ = 104.4 m

10. (a) $\dfrac{h}{AD} = \tan 40°$, $\therefore$ AD = $\dfrac{h}{\tan 40°}$. $\dfrac{h}{AC} = \tan 25°$, $\therefore$ AC = $\dfrac{h}{\tan 25°}$

(b) $\dfrac{h}{\tan 25°} - \dfrac{h}{\tan 40°} = 75$, $h\left(\dfrac{1}{\tan 25°} - \dfrac{1}{\tan 40°}\right) = 75$,

$h = 75 \div \left(\dfrac{1}{\tan 25°} - \dfrac{1}{\tan 40°}\right) = 79$. *Ans. The height of the cliff is 79 m.*

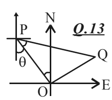
11. $\dfrac{PQ}{AQ} = \tan 38°$, AQ = $\dfrac{PQ}{\tan 38°}$; $\dfrac{PQ}{QB} = \tan 22°$, QB = $\dfrac{PQ}{\tan 22°}$;

$\dfrac{PQ}{\tan 38°} + \dfrac{PQ}{\tan 22°} = 120$, $PQ\left(\dfrac{1}{\tan 38°} + \dfrac{1}{\tan 22°}\right) = 120$,

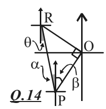
$PQ = 120 \div \left(\dfrac{1}{\tan 38°} + \dfrac{1}{\tan 22°}\right) = 32$. *Ans. The height of the lighthouse is 32 m.*

12. (a) a = 90° – 40° – 15° = 35°, b = 40° (alt. ∠s, // lines),
y = 45° (alt. ∠s, // lines), x = 40° + a (alt. ∠s, // lines) = 40° + 35° = 75°
(b) 40° + a = 40° + 35° = 75°, $\therefore$ the compass bearing of B from A is N75°E.
(c) 180° – y = 180° – 45° = 135°, $\therefore$ the true bearing of B from C is 135°.
(d) The true bearing of C from A is 040°.

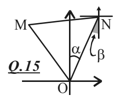
13. θ = 42°, θ + 39° = 42° + 39° = 81°.
Ans. Bearing of Q from P is S81°E.



14. β = 35°; α = 45° – β = 45° – 35° = 10°; θ = α = 10°.
Ans. Compass bearing of P from R is S10°E.



15. α = 60° – (360° – 318°) = 18°; β = α = 18°
$\therefore$ True bearing of M from N = 180° + β + 60° = 180° + 18° + 60° = 258°.



16. $\because$ ∠POQ = 65° + 25° = 90°, $\therefore$ tan ∠OQP = $\dfrac{24}{30}$, ∠OQP = 38.7°;

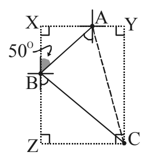
38.7° + 25° = 63.7°. *Ans. Compass bearing of P from Q is N63.7°W.*

17. OA = 80 × 2 = 160; OB = 60 × 2 = 120;

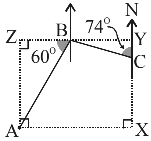
∠AOB = 360° – 325° + 55° = 90°, $\therefore$ AB = $\sqrt{160^2 + 120^2}$ = 200 km

Ans. Distance between A and B after 2 hours is 200 km.

18. AX = 1.5 sin 50°; BX = 1.5 cos 50°; BZ = 3 cos 50°; CZ = 3 sin 50°;
AY = CZ – AX = 3 sin 50° – 1.5 sin 50° = 1.149,
CY = BX + BZ = 1.5 cos 50° + 3 cos 50° = 2.893,
AC = $\sqrt{AY^2 + CY^2}$ = $\sqrt{1.149^2 + 2.893^2}$ = 3.1124 km = 3112.4 m
$\therefore$ Shortest time = 3112.4 ÷ 50 = 62.2 min.



19. (a) $BZ = 240\cos 60°$; $BY = 150\sin 74°$;



$\therefore$ $AX = BZ + BY = 240\cos 60° + 150\sin 74° = 264.19 = 264.2$

Ans. *He is 264.2 m west of the starting point.*

(b) $AZ = 240\sin 60°$; $CY = 150\cos 74°$;

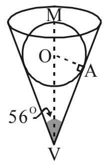
$\therefore$ $CX = AZ - CY = 240\sin 60° - 150\cos 74° = 166.5$

Ans. *He is 166.5 m south of the starting point.*

(c) Distance from the starting point = $\sqrt{AX^2 + CX^2}$

$= \sqrt{264.19^2 + 166.5^2} = 312.3$ m

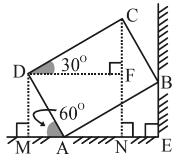
20. $OV = MV - MO = 8 - r$. In ΔOVA, $\dfrac{r}{8-r} = \sin(\dfrac{56°}{2})$,



$r = 8\sin 28° - r\sin 28°$, $r = \dfrac{8\sin 28°}{1+\sin 28°} = 2.56$.

Ans. *The radius of the sphere is 2.56 cm.*

21. (a) $DM = 20\sin 60° = 17.3$; $CF = 30\sin 30°$;



$CN = DM + CF = 20\sin 60° + 30\sin 30° = 32.3$.

Ans. *Distances from C and D to AE are 32.3 cm and 17.3 cm respectively.*

(b) $AM = 20\cos 60°$; $AE = 30\cos 30°$;

$ME = AM + AE = 20\cos 60° + 30\cos 30° = 36.0$

Ans. *Distance from D to BE is 36.0 cm.*

22. (a) Distance between H and K = $180 \times \dfrac{15}{60} = 45$ km

(b) Let x km be the perpendicular distance from A to HK.

$\therefore$ $\dfrac{x}{\tan 44°} + \dfrac{x}{\tan 79°} = 45$, $x = 45 \div \left(\dfrac{1}{\tan 44°} + \dfrac{1}{\tan 79°}\right) = 36.6$

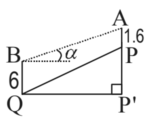
Ans. *The altitude of the helicopter is 36.6 km.*

23. (a) Let P'Q be the horizontal distance between P and Q.

$P'Q = 4 \times 400 = 1\,600$ m, $P'P = 300 - 100 = 200$ m,

$\therefore$ Actual length of road PQ = $\sqrt{200^2 + 1\,600^2} = 200\sqrt{65} = 1\,612.5$ m

(b) Gradient of road PQ = $\dfrac{200}{1\,600} = \dfrac{1}{8}$



Let θ be the angle of inclination, $\tan\theta = \dfrac{1}{8}$, $\theta = 7.13°$.

Ans. *The angle of inclination of road PQ is 7.13°.*

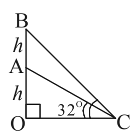
(c) Vertical distance between A and B = $200 + 1.6 - 6 = 195.6$ m,

horizontal distance = $P'Q = 1\,600$ m.

Let α be the angle of depression, $\tan\alpha = \dfrac{195.6}{1\,600}$, $\alpha = 6.97°$.

Ans. *The angle of depression from the man to the tree is 6.97°.*

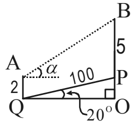
24. Let $2h$ be the height of the building. $\dfrac{h}{OC} = \tan 32°$;



$\tan\angle BCO = \dfrac{2h}{OCX} = 2\tan 32°$, $\therefore$ $\angle BCO = 51.3°$

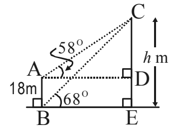
Ans. *The angle of depression from the top of building is 51.3°.*

25. Their horizontal distance = OQ = $100\cos 20°$; OP $=100\sin 20°$;
their vertical distance = OP + PB − AQ = $100\sin 20°+5-2$
$=100\sin 20° + 3$; $\tan\alpha = \dfrac{3+100\sin 20°}{100\cos 20°}$, $\alpha = 21.6°$

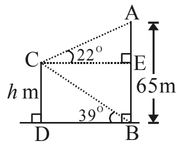


Ans. *The angle of depression required is 21.6°.*

26. Let h m be the height of the building, $\therefore$ CD = $(h-18)$ m.
In ΔBCE, BE = $\dfrac{h}{\tan 68°}$; in ΔACD, AD = $\dfrac{h-18}{\tan 58°}$. $\because$ BE = AD,
$\therefore \dfrac{h}{\tan 68°} = \dfrac{h-18}{\tan 58°}$, $h\tan 58° = h\tan 68° - 18\tan 68°$,
$h = \dfrac{18\tan 68°}{\tan 68° - \tan 58°} = 50.9$. *Ans.* *The height of the building is 50.9 m.*

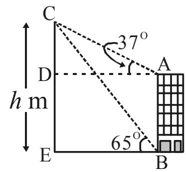


27. (a) In ΔACE, CE = $\dfrac{65-h}{\tan 22°}$; in ΔCDE, BD = $\dfrac{h}{\tan 39°}$.
$\because$ CE = BD, $\therefore \dfrac{65-h}{\tan 22°} = \dfrac{h}{\tan 39°}$,
$65\tan 39° - h\tan 39° = h\tan 22°$,
$\therefore h = \dfrac{65\tan 39°}{\tan 22° + \tan 39°} = 43.364 \approx 43.4$.



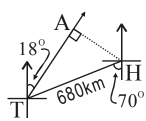
(b) Distance between two buildings = $\dfrac{43.364}{\tan 39°} = 53.6$ m

28. Let h m be the vertical height of the balloon,
$\therefore$ CD = $(h-80)$ m. In ΔACD, AD = $\dfrac{h-80}{\tan 37°}$;
in ΔBCE, BE = $\dfrac{h}{\tan 65°}$. $\because$ AD = BE, $\therefore \dfrac{h-80}{\tan 37°} = \dfrac{h}{\tan 65°}$,
$h\tan 65° - 80\tan 65° = h\tan 37°$, $h = \dfrac{80\tan 65°}{\tan 65° - \tan 37°} = 123.3$.



Ans. *The vertical height of the balloon is 123.3 m.*

29. $\dfrac{y}{\text{BE}} = \tan 45° = 1$, BE $= y$; AD = BE $= y$;
CD = AD $\tan\theta$, $\therefore y - x = y\tan\theta$, $y = \dfrac{x}{1-\tan\theta}$

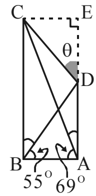


30. (a) Typhoon is nearest Hong Kong when TA ⊥ AH.
$\angle$ATH = 70° − 18° = 52°, $\dfrac{\text{AH}}{\text{TH}} = \sin 52°$,
AH = $680\sin 52°$ = 535.8 km.
Ans. *The shortest distance from Hong Kong is 535.8 km.*
(b) AT = $680\cos 52°$ = 418.65, $\therefore$ Time taken = 418.65 ÷ 160 = 2 h 37 min.
Ans. *It will be nearest Hong Kong at 2 h 37 min after 11:00 a.m., that is 1:37 p.m.*

31. CE = AB; BC= AB tan 69°; AE = BC = AB tan 69°; AD = AB tan 55°;

$\tan\theta = \dfrac{CE}{AE - AD} = \dfrac{AB}{AB\tan 69° - AB\tan 55°} = \dfrac{1}{\tan 69° - \tan 55°}$, $\theta = 40.4°$.

Ans. Bearing of C from D is N40.4°W.



32. (a) $PR = 60 \times \dfrac{20}{60} = 20$km; $\because \angle PQR = 180° - 50° - 40° = 90°$,

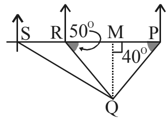
$\therefore QR = PR\cos 50° = 20\cos 50° = 12.9$ km

(b) $QM = QR\sin 50° = 20\cos 50°\sin 50° = 9.848$;

$MS = RM + RS = 20\cos 50°\cos 50° + 10 = 18.2635$;

$\tan\angle MSQ = \dfrac{QM}{MS} = \dfrac{9.848}{18.2635}$, $\angle MSQ = 28.3°$.

$\therefore$ True bearing of Q from S = 90° + 28.3° = 118.3°



(c) $SQ = \sqrt{MS^2 + QM^2} = \sqrt{18.2635^2 + 9.848^2} = 20.7$ km

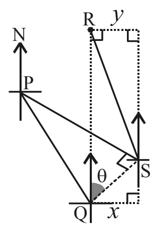
$\therefore$ Time taken = (20.7 ÷ 60) × 60 = 20.7 min

33. (a) The ship is nearest to Q when PS ⊥ SQ. In △PQS,

$\angle QPS = 60° - 32° = 28°$, $\therefore SQ = 40\sin 28° = 18.8$.

$\theta = 180° - 90° - 28° - 32° = 30°$.

Ans. At that instant the ship was 18.8 km in the direction of N30°E from Q.



(b) $\dfrac{x}{SQ} = \cos(90° - 30°)$, $x = (40\sin 28°)\cos 60°$;

$y = x$, $\therefore y = (40\sin 28°)\cos 60°$

$\dfrac{y}{RS} = \sin 20°$, $\therefore RS = \dfrac{(40\sin 28°)\cos 60°}{\sin 20°} = 27.453$ km

In △PQS, $\dfrac{PS}{PQ} = \cos 28°$, $PS = 40\cos 28° = 35.318$ km

$\therefore$ Time taken = (27.453 + 35.318) ÷ 28 = 2 h 15 min.

Ans. The ship reached R at 3:45 p.m.

34. Let $AB = a$; $BC = \sqrt{a^2 + a^2} = \sqrt{2}a$; $AC = \sqrt{a^2 + (\sqrt{2}a)^2} = \sqrt{3}a$

$\therefore \cos\angle ACB = \dfrac{BC}{AC} = \dfrac{\sqrt{2}a}{\sqrt{3}a} = \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$

35. ∠R = ∠P = 52° (base ∠s, isos. Δ); ∠AQD = 180° − 2(52°) = 76°

In ΔAQD, $\dfrac{AD}{AQ} = \sin 76°$, $AQ = \dfrac{6}{\sin 76°}$, ∠PBA = ∠R = 52° (corr. ∠s, AB // QR),

$\therefore$ PA = AB = 6 cm (sides opp. eq. ∠s). $\therefore PQ = PA + AQ = 6 + \dfrac{6}{\sin 76°} = 12.2$ cm

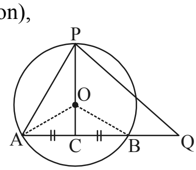
36. (a) In ΔOAC and ΔOBC, OA = OB (radii), OC = OC (common),

AC = BC (given), ∴ ΔOAC ≅ ΔOBC (S.S.S.),

∴ ∠OCA = ∠OCB (corr. ∠s, ≅ Δs).

∠OCA + ∠OCB = 180° (adj. ∠s on st. line),

∠PCA = ∠OCA = 180° ÷ 2 = 90°



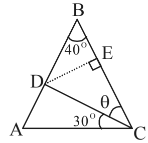
(b) AC = 8 ÷ 2 = 4, $OC = \sqrt{5^2 - 4^2} = 3$, PC = 5 + 3 = 8,

$\therefore AP = \sqrt{AC^2 + PC^2} = \sqrt{4^2 + 8^2} = 8.94$ cm

(c) $\dfrac{PC}{CQ} = \tan 56°$, $CQ = \dfrac{8}{\tan 56°} = 5.396$,

$\therefore$ Area of $\Delta APQ = \dfrac{1}{2} \times PC \times AQ = \dfrac{1}{2}(8)(4 + 5.396) = 37.6 \text{ cm}^2$

37. $\angle BAC = \angle BCA$ (base $\angle$s, isos. Δ),



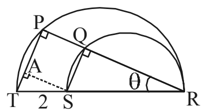
$\therefore$ $\theta + 30° = \dfrac{180° - 40°}{2}$ ($\angle$ sum of Δ), $\theta = 40°$,

$\therefore$ CD = BD (sides opp. eq. $\angle$s),

$\therefore$ $\Delta DCE \cong \Delta DBE$ (AAS). CE = BE (corr. sides, $\cong \Delta$s),

$CE = \dfrac{1}{2} BC = 6$. In ΔDCE, $\dfrac{CE}{CD} = \cos 40°$, $\therefore$ $CD = \dfrac{6}{\cos 40°} = 7.83\text{cm}$

38. Draw SA $\perp$ PT, $\therefore$ APQS is a rectangle.



$\therefore$ $\angle AST = \theta$ (corr. $\angle$s, AS // PR).

In ΔSTA, $\dfrac{AS}{TS} = \cos\theta$, $AS = 2\cos\theta$.

$\because$ PQ = AS (rectangle property), $\therefore$ $PQ = 2\cos\theta$

39. (a) In ΔEAB and ΔFBE, $\angle A = \angle B$ (prop. of square);

$\angle ADE + 90° = \angle DEB$ (ext. $\angle$ of Δ), $\angle ADE + 90° = 90° + \angle BEF$,

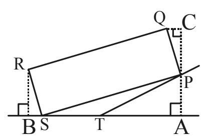
$\therefore$ $\angle ADE = \angle BEF$; $\angle AED = \angle BFE$ (3^{rd} $\angle$ of Δ); $\therefore$ $\Delta EAD \sim \Delta FBE$ (AAA)

(b) Let the side of the square be x.

$\because$ $\Delta FBE \sim \Delta EAD$ (proved), $\therefore$ $\dfrac{BF}{AE} = \dfrac{BE}{AD}$, $\dfrac{BF}{\frac{1}{2}x} = \dfrac{\frac{1}{2}x}{x}$, $BF = \dfrac{1}{4}x$,

$\therefore$ $CF = x - \dfrac{1}{4}x = \dfrac{3}{4}x$, $\therefore$ $\tan\theta = \dfrac{CF}{CD} = \dfrac{\frac{3}{4}x}{x} = \dfrac{3}{4}$

40. In ΔAPT, $\dfrac{PA}{PT} = \sin 28°$, $PA = 12\sin 28°$.



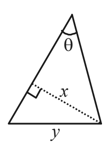
In ΔPSA, $\sin\angle PSA = \dfrac{PA}{PS} = \dfrac{12\sin 28°}{18}$, $\angle PSA = 18.24°$;

$\angle BRS = \angle PSA = 18.24°$. In ΔRBS, $\dfrac{RB}{6} = \cos 18.24°$;

$\therefore$ Height of R = RB = 6 cos 18.24° = 5.70 cm

$\therefore$ Height of Q = AC = CP + PA = RB + PA = 5.70 + 12 sin 28° = 11.3 cm

41. (a) $\theta = 180° - 60° - 75° = 45°$ ($\angle$ sum of Δ);



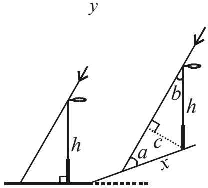
$\dfrac{x}{8} = \sin 45°$, $x = 8\sin 45°$; $\dfrac{x}{y} = \sin 60°$,

$\therefore$ $y = \dfrac{x}{\sin 60°} = \dfrac{8\sin 45°}{\sin 60°} = 6.53$

(b) $a = 60° - 20° = 40°$,
$b = 180° - 90° - 60° = 30°$ ($\angle$ sum of Δ)

(c) $\dfrac{h}{2.3} = \tan 60°$, $h = 2.3\tan 60° = 3.98$

Ans. The height of the lamppost is 3.98 m.



(d) $\frac{c}{h} = \sin b$, $c = 2.3 \tan 60° \sin 30°$; $\frac{c}{x} = \sin a$,

$\therefore\ x = \frac{2.3 \tan 60° \sin 30°}{\sin 40°} = 3.10$

Ans. The length of shadow on the slope is 3.10 m.

42. (a) $\angle BAC = 185° - 95° = 90°$

$\angle ABC = 95° - (234° - 180°) = 41°$

(b) Vertical distance between A and C = 350 − 150 = 200 m.

Horizontal distance between A and C = 3 × 4 000 cm = 12 000 cm = 120 m.

Let θ be the angle of elevation of A from C.

$\tan\theta = \frac{200}{120}$, $\theta = 59.0°$ (3 sig. fig.)

Ans. The angle of elevation of A from C is 59.0°.

(c) (i) Let x m be the horizontal distance between A and B.

From (a), $\angle BAC = 90°$, $\angle ABC = 41°$

$\tan 41° = \frac{120}{x}$, $x = \frac{120}{\tan 41°} = 138$ (3 sig. fig.)

Ans. The required distance is 138 m.

(ii) The vertical distance between A and B = 400 − 350 = 50 m.

Let α be the angle elevation of B from A.

$\tan\alpha = \frac{50}{138}$, $\alpha = 19.9°$ (3 sig. fig.)

Ans. The required angle of elevation is 19.9°.

43. (a) Distance between Q and R = 0.808 (6 × 60 + 15) = 0.808 (375) = 303 m.

Vertical distance between Q and R = 420 − 360 = 80 m.

Horizontal distance between Q and R = $\sqrt{303^2 - 60^2}$ = 297 m.

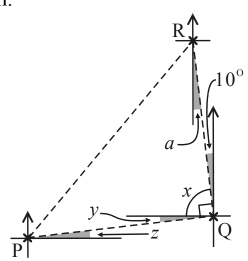
Scale of the map = 9.9 cm : 297 m = 9.9 : 29 700 = 1:3 000.

(b) With the notations in the figure,

$x = 90° - 10° = 80°$, $y = 90° - x = 90° - 80° = 10°$

$z = y = 10°$, $90° - z = 90° - 10° = 80°$

Ans. The require compass bearing is N80°E.



(c) $\cos\angle QRP = \frac{9.9}{13.3}$, $\therefore\ \angle QRP = 41.9°$.

$41.9° - a = 41.9° - 10° = 31.9°$.

Ans. The compass bearing of P from R is S31.9°W.

44. (a) (i) $\angle PQR = 46° + 68° = 114°$

$\because$ PQ = QR , $\therefore\ \angle QPR = \angle QRP$ (base ∠s, isos. Δ)

$\angle QPR + \angle QRP + \angle PQR = 180°$ (∠ sum of Δ)

$2\ \angle QPR + 114° = 180°$, $\angle QPR = 33°$, $46° + 33° = 79°$

Ans. The compass bearing of R from P is N79°E.

(ii) Let M be the mid-point of PR. Note that QM ⊥ PR.

$\frac{PM}{PQ} = \cos \angle QPR$, PM = 6 cos 33°,

PR = 2 PM = 2 (6 cos 33°) = 12 cos 33° = 10.1 km (3 sig. fig.)

(b) (i) Let S′ be a point on PR such that SS′ ⊥ PR. SS′ = h km.

$\angle PSS' = 90° - 64° = 26°$, $\angle RSS' = 90° - 25° = 38°$,

$\frac{PS'}{SS'} = \tan \angle PSS'$, $PS' = h \tan 26°$; $\frac{RS'}{SS'} = \tan \angle PSS'$, $RS' = h \tan 38°$

PS′ + RS′ = PR , $h \tan 26° + h \tan 38° = 12 \cos 33°$

$h (\tan 26° + \tan 38°) = 12 \cos 33°$

$h = \frac{12 \cos 33°}{\tan 26° + \tan 38°} \approx 7.930577 = 7.93$ (3 sig. fig.)

(ii) Let α be the angle of depression of Q from the helicopter.

α is greatest when the helicopter is above M.

$\tan \alpha = \frac{7.930577}{6 \sin 33°}$, α = 67.6° > 65° (3 sig. fig.)

∴ The claim is disagreed.

45. (a) AB = 80 × 1.5 = 120 km

$\angle BAP = 300° - 270° = 30°$, $\angle CAP = 270° - 225° = 45°$

BP = AB sin ∠BAP = 120 sin 30° = 60 km

AP = AB cos ∠BAP = 120 cos 30° = $60\sqrt{3}$ km

PC = AP tan ∠CAP = $60\sqrt{3} \tan 45° = 60\sqrt{3}$ km

BC = BP + PC = 60 + $60\sqrt{3}$ = 60 (1 + $\sqrt{3}$) km

(b) $AC = \frac{PC}{\sin \angle CAP} = \frac{60\sqrt{3}}{\sin 45°} = 60\sqrt{6}$ km

∴ Exact speed of car Q = $\frac{60\sqrt{6}}{1.5} = 40\sqrt{6}$ km/h

(c) $\angle BCD = 360° - 323° = 37°$

BD = BC tan ∠BCD = 60 (1 + $\sqrt{3}$) tan 37°

Time for car P to travel from B to D

= 60 (1 + $\sqrt{3}$) tan 37° ÷ 80 = 1.54 h (3 sig. fig.)

$CD = \frac{BC}{\cos \angle BCD} = \frac{60(1+\sqrt{3})}{\cos 37°}$

Time for car Q to travel from C to D = $\frac{60(1+\sqrt{3})}{\cos 37°} \div 40\sqrt{6}$ = 2.09 h (3 sig. fig.)

1.54 h < 2.09 h , ∴ the claim is agreed.

Unit 15 Measures of central tendency

1. (a) Mean $=\frac{49}{8}=6.125$, Mode = 8, Median $=\frac{6+8}{2}=7$

(b) −14 , – 11, −11, – 10, −7; Mean $=\frac{-53}{5}=-10.6$, Mode = −11, Median = −11

(c) −18, −13, −10, 13, 15, 18; Mean = $\frac{5}{6}$, No mode, Median = $\frac{-10+13}{2}=1.5$

(d) $-8°C,\ -5°C,\ -4°C,\ -1°C,\ 0°C,\ 0°C,\ 2°C$;

Mean $=\frac{-16}{7}=-2\frac{2}{7}°C$, Mode = 0°C, Median = −1°C

(e) $2.4\text{ m}^2,\ 6.2\text{ m}^2,\ 7\text{ m}^2,\ 9.4\text{ m}^2,\ 10\text{ m}^2,\ 11.5\text{ m}^2,\ 12\text{ m}^2$;

Mean $=\frac{58.5}{7}=8.36\text{ m}^2$, No mode, Median = 9.4 m^2

(f) 20.4, 23.2, 24.1, 24.3, 24.7, 24.7, 28.5;

Mean $=\frac{169.9}{7}=24.3°C$, Mode = 24.7° C, Median = 24.3°C

(g) $x-2, x-1, x, x, 3x, 5x$; Mean $=\frac{12x-3}{6}$, Mode = x, Median = $\frac{x+x}{2}=x$

(h) 60 cm, 0.6m, 75 cm, 1 m, 1.3 m;

Mean $=\frac{425}{5}=85\text{ cm}$, Mode = 60 cm, Median = 75 cm

2. (a) Mean $=\frac{12\times12+13\times10+14\times15+15\times13}{12+10+15+13}=\frac{679}{50}=13.58$, Median = 14, Mode = 14

(b) Mean $=\frac{1\times6+2\times15+3\times16+4\times1+5\times2}{6+15+16+1+2}=\frac{98}{40}=2.45$, Median = 2, Mode = 3

3. (a) The class marks are 145, 245, 345, 445 and 545.

Total number of shoes = 7 + 16 + 33 + 24 + 20 = 100;

∴ Mean = $(145\times7+245\times16+345\times33+445\times24+545\times20)\div100$

= 37900 ÷ 100 = $379

(b) The modal class is $300 – $390.

4. (a)

Weight (kg)	30 – 34	35 – 39	40 – 44	45 – 49	50 – 54
Frequency	3	14	9	4	6

(b) The modal class is 35 kg – 39 kg.

5. (a) Weight mean score of Apple $=\frac{72\times3+85\times3+74\times2}{3+3+2}=\frac{619}{8}=77.4$

Weight mean score of Banana $=\frac{64\times3+87\times3+76\times2}{8}=\frac{605}{8}=75.6$

Weight mean score of Cherry $=\frac{74\times3+67\times3+84\times2}{8}=\frac{591}{8}=73.9$

(b) Apple achieved the best result.

6. (a) The modal class is 161 cm – 165 cm.

(b) Mean $= \dfrac{153\times3+158\times9+163\times14+168\times12+173\times2}{40} = \dfrac{6525}{40} \approx 163.1\,\text{cm}$

7. (a) A = 25, B = 11500

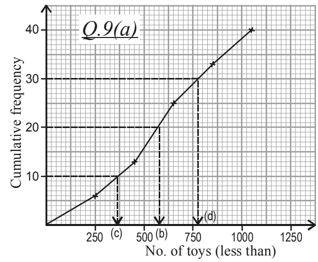
(b) Median salary = $11500

8. The class marks are
14.5, 24.5, 34.5, 44.5, 54.5, 64.5 and 74.5.
Mean = (14.5×18+24.5×11+34.5×15+ 44.5×26 +54.5×22+64.5×30+74.5×28) ÷ 150 = 7425 ÷ 150 = 49.5

9. (b) From the graph, the median daily production is 575.

(c) From the graph, the lower quartile is 375.

(d) From the graph, the upper quartile is 775.



10. (a)

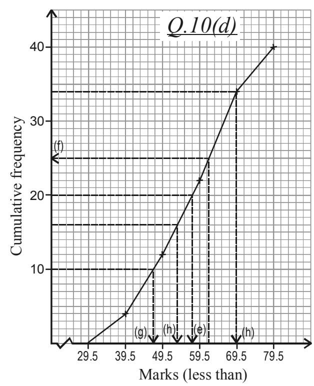
Marks	Class boundaries	Class mark(x)	Frequency (f)	fx
30 – 39	29.5 – 39.5	34.5	4	138
40 – 49	39.5 – 49.5	44.5	8	356
50 – 59	49.5 – 59.5	54.5	10	545
60 – 69	59.5 – 69.5	64.5	12	774
70 – 79	69.5 – 79.5	74.5	6	447
		Total:	40	2260

(b) Mean mark $= \dfrac{2260}{40} = 56.5$.

The modal class is 60 – 69.

(c)

Marks less than	Cumulative frequency
39.5	4
49.5	12
59.5	22
69.5	34
79.5	40



(e) From the graph, the median mark is 57.5.

(f) Percentage $= \dfrac{40-25}{40}\times100\% = 37.5\%$

(g) 75% of the students pass means 25% of the students fail, from the graph, the passing mark is 47.

(h) $40\times40\% = 16$, from the graph, the 40$^{\text{th}}$ percentile is 53.5.

$40\times85\% = 34$, from the graph, the 85$^{\text{th}}$ percentile is 69.5.

11. Mean fare $= \dfrac{11.9\times12+6.9\times8}{20} = \dfrac{198}{20} = \$9.9$

12. $\dfrac{3+x+2x+(x+6)+11}{5} = 8$, $4x+20=40$, $\therefore\ x=5$

13. Salary in December = 9 000 × 12 – 8 500 × 11= $14 500

14. Mean $= \dfrac{a+b+c+d+17+19}{6} = \dfrac{15\times4+17+19}{6} = \dfrac{96}{6} = 16$

15. Mean weight $= \dfrac{48\times25+35\times15}{40} = \dfrac{1725}{40} = 43.125$ kg

16. The last number = 14 × 10 – 15 × 9 = 5

17. Correct mean mark $= \frac{65 \times 40 - 17 + 71}{40} = \frac{2654}{40} = 66.35$

18. (a) Mean age = 22 + 9 = 31

(b) Mean age of the remaining 10 members $= \frac{31 \times 11 - 36}{10} = \frac{305}{10} = 30.5$

19. $\because$ x is the median, $\therefore$ $6 \le x \le 8$, $\therefore$ possible values of x are 6, 7 and 8.

20. $\frac{17 + 20 + y + 27 + 31 + 36}{6} = \frac{y + 27}{2}$, $\frac{131 + y}{6} = \frac{y + 27}{2}$,

$131 + y = 3y + 81$, $2y = 50$, $\therefore$ $y = 25$

21. $\frac{3 + x + 8 + 9}{4} = 6.5$, $20 + x = 26$, $\therefore$ $x = 6$. $\frac{3 \times 5 + 6 \times 6 + 8 \times 9 + 9 \times y}{5 + 6 + 9 + y} = 7.1$,

$\frac{123 + 9y}{20 + y} = 7.1$, $123 + 9y = 142 + 7.1y$, $1.9y = 19$, $\therefore$ $y = 10$

22. (a) 2, 4, 5, 6, 6, 10, 12, 15; Mean $= \frac{60}{8} = 7.5$, Mode = 6, Median = $\frac{6 + 6}{2} = 6$

(b) (i) Mean $= \frac{7.5 \times 8 + 12}{9} = 8$, Mode = 6 and 12, Median = 6

(ii) Mean $= \frac{7.5 \times 8 - 4}{7} = 8$, Mode = 6, Median = 6

(iii) Mean = 7.5 + 4 = 11.5, Mode = 6 + 4 = 10, Median = 6 + 4 = 10

(iv) Mean = 7.5 × 2 = 15, Mode = 6 × 2 = 12, Median = 6 × 2 = 12

23. $\frac{x}{6}, \frac{x}{2}, \frac{2x}{3}, \frac{3x}{4}, \frac{6x}{5}$; $\frac{2x}{3} = 10$, $\therefore$ $x = 15$

24. $\frac{4 + x + y + 7}{4} = 6$, $x + y + 11 = 24$, $y = 13 - x$; $\frac{3 + y + 2x}{3} = 7$, $3 + (13 - x) + 2x = 21$,

$16 + x = 21$, $\therefore$ $x = 5$, $\therefore$ $y = 13 - 5 = 8$

25. $\frac{k \times n - 3 - 7 - 12}{n - 3} = k$, $nk - 22 = nk - 3k$, $3k = 22$, $\therefore$ $k = \frac{22}{3}$

26. Median = 20 × 9 − 25 × 4 − 16 × 4 = 16

27. $a + b + c = 12 \times 5 - 9 \times 2 = 42$, but $b = a + 2$ and $c = a + 4$,

$\therefore$ $a + (a + 2) + (a + 4) = 42$, $3a + 6 = 42$, $3a = 36$, $\therefore$ $a = 12$,

$b = 12 + 2 = 14$ and $c = 12 + 4 = 16$

28. (a) $x - 9, x - 5, x + 3, x + 7, x + 9$; Median = $x + 3$

(b) $\frac{(x - 9) + (x - 5) + (x + 3) + (x + 7) + (x + 9)}{5} = \frac{x + 3}{2}$, $\frac{5x + 5}{5} = \frac{x + 3}{2}$,

$2x + 2 = x + 3$, $\therefore$ $x = 1$

29. $\because$ Mode = 33 and $p < q$, $\therefore$ $p = 33$.

$\because$ Median = 37, $\therefore$ $\frac{q + 39}{2} = 37$, $q + 39 = 74$, $\therefore$ $q = 35$

30. (a) Mean mark = $\frac{76 \times 40 + 58 \times 32}{72} = \frac{4\,896}{72} = 68$

(b) Correct mean mark = $\dfrac{76\times39+58\times32-48+84}{72}=\dfrac{4\,856}{72}=67.4$

31. (a) $\dfrac{63\times1+84\times2+77\times1+n\times3}{1+2+1+3}=83$, $\dfrac{308+3n}{7}=83$, $308+3n=581$,

$3n=273$, $\therefore\ n=91$

(b) The new result = $\dfrac{63\times1+84\times2+77\times1+91\times4}{1+2+1+4}=84$,

$\therefore$ Percentage change = $\dfrac{84-83}{83}\times100\%$ = 1.20% (increase)

32. (a) $15+32+x+y+6+4=100$, $y=43-x$;

$\dfrac{0\times15+1\times32+2x+3y+4\times6+5\times4}{100}=1.75$, $2x+3y+76=175$,

$2x+3(43-x)=99$, $-x=-30$, $\therefore\ x=30$, $\therefore\ y=43-30=13$

(b) Modal number = 1

(c) Median number = 2

33. (a) 32, 36, 40, 48, 52, 60; Mean = $\dfrac{268}{6}=44\dfrac{2}{3}$, Median = $\dfrac{40+48}{2}=44$

(b) Let x be the number removed, $\dfrac{268-x}{5}=44\dfrac{2}{3}-\dfrac{22}{15}=\dfrac{216}{5}$, $268-x=216$,

$x=52$. *Ans. The removed number is 52.* $\therefore$ The new median is 40.

34. $\because$ Mode = 16, $\therefore\ r+1=16$, $\therefore\ r=15$.

$\dfrac{(p+1)+q}{2}=11$, $p+1+q=22$, $p=21-q$

$\dfrac{4+6+6+p+(p+1)+(p+1)+q+q+15+16+16+16}{12}=11$, $3p+2q+81=132$,

$\therefore\ 3(21-q)+2q=51$, $-q=-12$, $\therefore\ q=12$, $\therefore\ p=21-12=9$

35. (a) Median = y = 36; $\dfrac{x}{36}=\dfrac{3}{4}$, $\therefore\ x=27$; $\dfrac{z}{36}=\dfrac{2}{4}$, $\therefore\ z=18$

(b) Let $x=3k$, $y=4k$, $z=2k$, $\dfrac{3k+3k+4k+2k+2k}{5}=8.4$

$14k=42$, $k=3$, $\therefore\ x=9$, $y=12$, $z=6$

36. (a) Every datum is multiplied by 3, $\therefore$ Mean = $3x$, Mode = $3y$, Median = $3z$

(b) Each number of data is trebled, $\therefore$ Mean = x, Mode = y, Median = z

(c) Every datum is multiplied by –1 and then plus 4,

$\therefore$ Mean = $4-x$, Mode = $4-y$, Median = $4-z$

37. $x+10$ and $x+16$ are the greatest, and $4-x<8-x$.

$\because\ 8-x$ is the median, $\therefore\ 8-x\ge x$, $2x\le8$, $x\le4$

Since x is positive, $\therefore\ x$ = 1, 2, 3 or 4

38. (a) $18-n>15$, $n<3$ … (i);

$18-n>n+1$, $17>2n$, $n<8.5$ … (ii)

Combining the two cases, $n<3$. *Ans. Possible values of n are 0, 1 and 2.*

(b) $n+1>15$, $n>14$ … (i);

$n+1>18-n$, $2n>17$, $n>8.5$ … (ii);

$18-n\geq 0$, $n\leq 18$ … (iii) Combining the three cases, $14<n\leq 18$.

Ans. Possible values of n are 15, 16, 17 and 18.

(c) If median is 14, $10+18-n>n+1+15$, $12>2n$, $n<6$

On the other hand, mode = 14 when n = 0, 1 or 2.

∴ It is possible for both the median and modal age to be 14.

39. (a) Mean = $\dfrac{8\,400+10\,000+14\,000+20\,000+25\,000+28\,000+120\,000}{8}=\$28\,175$,

Median = $\dfrac{14\,000+20\,000}{2}=\$17\,000$

(b) Median is a better measure of central tendency since it is not affected by the extreme data ($0 and $120 000).

40. (a) Extreme data exist and these data are much higher than the average.

(b) Extreme data exist and these data are much lower than the average.

(c) If Class A is chosen, the reason should be: (1) there are some very able students in Class A; or (2) there are some very weak students in Class B. If Class B is chosen, the reason should be: the difference between the mean and the median is smaller in Class B, and therefore the difference among students are not so great in Class B as in Class A.

41. $120(55)-3x=120(55)\left(1-1\frac{9}{11}\%\right)$, $3x=120(55)\left(1\frac{9}{11}\%\right)$, $3x=120$, $x=40$

42. (a) (i) $\dfrac{9(0)+6(1)+n(2)+4(3)+3(4)}{9+6+n+4+3}=1.44$,

$30+2n=1.44(22+n)$, $0.56n=1.68$, $n=3$

(ii) Number of students of Class 3A = 9 + 6 + 3 + 4 + 3 = 25

median = 1, mode = 0

(b) Let m be the mean number of hikes of Class 3B.

$\dfrac{25(1.44)+30m}{25+30}>2.1$, $36+30m>115.5$, $30m>79.5$, $m>2.65$

Ans. The mean number of hikes of Class 3B was greater than 2.65.

43. (a) $k=8$

(b) (i) $7+6+3=16$

$k<9+16$, $k<25$ *Ans. The greatest value of k is 24.*

(ii) $k+9>16$, $k>7$ *Ans. The least value of k is 8.*

(c) (i) $2k+9(3)+7(4)+6(5)+3(6)=3.06\,(k+9+7+6+3)$,

$2k+103=3.06k+76.5$, $1.06k=26.5$, $k=25$

(ii) Total no. of teenagers = $k+9+16=25+25=50$

Median $=\dfrac{\text{25th datum + 26th datum}}{2}=\dfrac{2+3}{2}=2.5$

44. (a) Mean $=\dfrac{173+175+178+182+184+187(2)+189+190+191(2)+193}{12}=\dfrac{2220}{12}=185$ cm

Median = 187 cm. Mode = 187 cm and 191 cm

(b) (i) New mean height $= \dfrac{2220+4(188)}{12+4} = 185.75$ cm

(ii) $a + b + 188 + 189 = 4(188)$, $a + b = 375$

(iii) 188 cm and 189 cm are both greater than the original median 187 cm.

For the new median to be 187 cm, the necessary condition is:

$a \le 187$ and $b \le 187$, $a + b \le 187 + 187 = 374$, which contradict the result of (b)(ii).

Thus, it is impossible.

45. (a) $\dfrac{375+377+(380+a)+382+(380+b)+391}{6} = 382$, $2\,285 + a + b = 2\,292$, $a + b = 7$

Note that $6 \le b \le 8$, and $0 \le a \le 2$.

For $b = 6$, $a = 7 - 6 = 1$. For $b = 7$, $a = 7 - 7 = 0$. For $b = 8$, $a = 7 - 8 = -1$ (rejected)

Ans. $a = 1$ and $b = 6$; or $a = 0$ and $b = 7$.

(b) Median = 396 Mbps

(c) (i) New median = 396(1 + 20%) = 475.2 Mbps

(ii) The sum of the 13 known data = 4756

Original mean $= \dfrac{4\,756+(380+a)+(380+b)+(390+b)}{15} = \dfrac{5906+a+2b}{15}$

$a + 2b$ is greatest when $a = 0$ and $b = 7$,

$\therefore$ the greatest original mean $= \dfrac{5\,906+0+2(7)}{15} = 394\dfrac{2}{3}$ Mbps

The greatest new mean $= 394\dfrac{2}{3}(1+20\%) = 473.6$ Mbps < 475.2 Mbps

Thus, the claim is disagreed.

Unit 16 Introduction to probability

1. P(not W) = $\dfrac{8+6}{10+8+6} = \dfrac{14}{24} = \dfrac{7}{12}$

2. Favourable outcomes: 3, 6, 9, … 27, 30; $\therefore$ P(correct date) = $\dfrac{10}{31}$

3. P(good apple) = $\dfrac{100-16}{100-1} = \dfrac{84}{99} = \dfrac{28}{33}$

4. P(red or Queen) = $\dfrac{26+2}{52} = \dfrac{28}{52} = \dfrac{7}{13}$

5. (a) Favourable outcomes: 6, 12, 18, 24, 30; $\therefore$ P(even and multiple of 3) = $\dfrac{5}{30} = \dfrac{1}{6}$

(b) Favourable outcomes: 1 – 9, 11, 13, 17, 19, 23, 29;

$\therefore$ P(prime number or smaller then 10) = $\dfrac{15}{30} = \dfrac{1}{2}$

(c) P(an integer) = $\dfrac{30}{30} = 1$ (d) P(divisible by 40) = $\dfrac{0}{30} = 0$

(e) Favourable outcomes: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30. $\therefore$ P(factor of 60) = $\dfrac{11}{30}$

6. Let n be the number of \$2 coins. $\frac{18}{18+n}=\frac{2}{3}$, $54=36+2n$, $18=2n$, $n=9$

   Ans. The number of \$2 coins is 9.

7. P(2nd one is \$2 coin) = $\frac{3}{7+3-1}=\frac{3}{9}=\frac{1}{3}$

8. Number of possible outcomes = 2 × 2 × 2 = 8;

   Favourable outcomes: BBB, BBG, BGB, GBB; $\therefore$ P(not more than 1 girl) = $\frac{4}{8}=\frac{1}{2}$

9. (a) P(6) = $\frac{20}{500}=\frac{1}{25}$ (b) P(smaller than 4) = $\frac{100+160+140}{500}=\frac{400}{500}=\frac{4}{5}$

10. Number of green balls = $12\times\frac{51}{200}=3.06\approx 3$

    Number of blue balls = $12\times\frac{69}{200}=4.14\approx 4$

    Number of white balls = $12\times\frac{80}{200}=4.8\approx 5$

11. Number of possible outcomes = 2 × 2 × 2 = 8.

    (a) Favourable outcomes: HTT, THT, TTH; $\therefore$ P(1H and 2T) = $\frac{3}{8}$.

    (b) Favourable outcome: TTT; $\therefore$ P(no H) = $\frac{1}{8}$

12. (a) Number of possible outcomes = 6 × 6 = 36

    Favourable outcomes: (5,5), (5,6), (6,5), (6,6); $\therefore$ P(both not less than 5) = $\frac{4}{36}=\frac{1}{9}$

    (b) Favourable outcomes: (1,1),(2,2), (2,1), (3,2), (3,3), (3,1), ….., (6,6), (6,5), (6,4), (6,3), (6,2), (6,1); $\therefore$ P(1st no. not smaller than 2nd no.) = $\frac{21}{36}=\frac{7}{12}$

    (c) Unfavourable outcomes: (1,3), (2,2), (2,6), (3,1), (3,5), (4,4), (5,3), (6,2), (6,6);

    $\therefore$ P(sum is not multiple of 4) = $\frac{36-9}{36}=\frac{27}{36}=\frac{3}{4}$

    (d) Favourable outcomes: (1,4), (2,4), (3,4), (5,4), (6,4), (4,1), (4,2), (4,3), (4,5), (4,6);

    $\therefore$ P('4' occurs exactly once) = $\frac{10}{36}=\frac{5}{18}$

    (e) Favourable outcomes: (1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)

    $\therefore$ P(product is odd) = $\frac{9}{36}=\frac{1}{4}$

    (f) Favourable outcomes: (2,3), (3,2); $\therefore$ P('2' and '3') = $\frac{2}{36}=\frac{1}{18}$

    (g) Favourable outcomes: (1,2), (2,1), (2,3), (3,2), ….. , (5,4), (5,6), (6,5);

$\therefore$ P(difference =1) = $\dfrac{10}{36}=\dfrac{5}{18}$

13. His expected age = $13\times\dfrac{5}{36}+14\times\dfrac{20}{36}+15\times\dfrac{11}{36} = 14\dfrac{1}{6}$ = 14.2 years old.

14. Favourable outcomes: (O,F), (F,O), (R,O), (O,R), (T,O), (O,T)

    No. of possible outcome = 4 × 3 = 12, $\therefore$ P(meaningful English word) = $\dfrac{6}{12}=\dfrac{1}{2}$

15. Favourable outcomes: (49, 52), (49, 60), (52, 49), (60, 52), (52, 60), (60, 49)

    No. of possible outcomes = 4 × 3 = 12, $\therefore$ P(exceed 100 kg) = $\dfrac{6}{12}=\dfrac{1}{2}$

16. (a) There are 20 possible outcomes.

    Favourable outcomes: (2,3), (2,9), (2,11), (3,2), (9,2), (11,2), (11,12), (12,11);

    $\therefore$ P(prime number) = $\dfrac{8}{20}=\dfrac{2}{5}$

    (b) Favourable outcomes: (2,3), (3,2), (3,12), (9,11), (11,9), (12,3);

    No. of possible outcomes = 5 × 4 = 20, $\therefore$ P(divisible by 5) = $\dfrac{6}{20}=\dfrac{3}{10}$

17. (a) P (6-mark region) = $\dfrac{5^2}{40^2} = \dfrac{1}{64}$, P (3-mark region) = $\dfrac{20^2-5^2}{40^2} = \dfrac{15}{64}$

    P (1-mark region) = $\dfrac{40^2-20^2}{40^2} = \dfrac{3}{4}$

    (b) The expected score = $6\times\dfrac{1}{64}+3\times\dfrac{15}{64}+1\times\dfrac{3}{4} = 1\dfrac{35}{64}$ = 1.55

18. Number of possible outcomes = 10 × 10 × 10 = 1 000,

    $\therefore$ P(open the safe in 1st trial) = $\dfrac{1}{1000}$

19. Let n be the number of girls,

    $\therefore$ P(getting a girl) = $\dfrac{n}{n+55\%n}=\dfrac{n}{1.55n}=\dfrac{100}{155}=\dfrac{20}{31}$

20. Let n be the total number of students,

    $\therefore$ P(boy not wearing glasses) = $\dfrac{n\times 40\%\times(1-30\%)}{n}=\dfrac{40}{100}\times\dfrac{70}{100}=\dfrac{7}{25}$

21. Number of students passed both subjects = $200-(200\times 27\%+200\times 32\%-16)=98$,

    $\therefore$ P(the student passed both subjects) = $\dfrac{98}{200}=\dfrac{49}{100}$

22. The possible position of the yellow ball: YOOO, <u>OYOO</u>, <u>OOYO</u>, OOOY

    $\therefore$ P(at least one orange ball separated by yellow ball) = $\dfrac{2}{4}=\dfrac{1}{2}$

23. P(get a marked fish) = $\dfrac{50}{x}=\dfrac{4}{50}$, $4x=2\,500$, $x=625$.

Ans. The approximate number of fish is 625.

24 (a) Favourable outcomes: 3 004, 3 008, 3 012, …., 3 096, 3 100;

$\therefore$ P(multiple of 4) = $\frac{25}{100} = \frac{1}{4}$

(b) Favourable outcomes: 3 003, 3 006, 3 009, …., 3 096, 3 099;

$\therefore$ P(multiple of 3) = $\frac{33}{100}$

(c) There are 8 multiples of 12: 3 012, 3 024, 3 036, …., 3 096;

$\therefore$ P(either multiple of 3 or multiple of 4) = $\frac{25+33-8}{100} = \frac{50}{100} = \frac{1}{2}$

25. Let n be the number of white balls. $\because$ P(W) = $\frac{4}{9} < \frac{1}{2}$,

$\therefore$ the no. of black balls = $n+4$. $\frac{n}{n+n+4} = \frac{4}{9}$, $9n = 8n+16$, $n = 16$,

$\therefore$ Total number of balls = $16+16+4 = 36$

26. The last 2 digits must be a number divisible by 4.

Favourable outcomes: 0, 4, 8. $\therefore$ P(divisible by 4) = $\frac{3}{10}$

27. Assume the seat of B is fixed.

Number of possible seats of A = 4, number of seats not next to B = 2

$\therefore$ P(A doesn't sit next to B) = $\frac{2}{4} = \frac{1}{2}$

28. Number of possible outcomes = 4 × 4 = 16.

Favourable outcomes: ES, SE (E: east, S: south).

$\therefore$ P(reach (–1, 2) after 2 moves) = $\frac{2}{16} = \frac{1}{8}$

29. Number of favourable outcomes = 3 × 4 = 12.

There are 2 unfavourable outcomes: (R, R), (R,R)

$\therefore$ P(different colours) = $\frac{12-2}{12} = \frac{10}{12} = \frac{5}{6}$

30. Favourable outcomes: 101, 102, …., 109; 110, 120, …, 190,

201, 202, …., 209; 210, 220, …., 290,

……

901, 902, …., 909; 910, 920, …., 990;

$\therefore$ P(exactly one digit is 0) = $\frac{9\times9+9\times9}{900} = \frac{162}{900} = \frac{9}{50}$

31. (a) No. of possible outcomes = 2× 2 × 2 = 8.

Unfavourable outcomes: all go to restaurant B , $\therefore$ P(at least one go to A) = $\frac{8-1}{8} = \frac{7}{8}$

(b) P(same restaurant) = $\frac{2}{8} = \frac{1}{4}$

32. Let the envelopes and the letters for Adam, Benjamin and Cart be E_A, E_B, E_C and L_A, L_B, L_C respectively.
Possible outcomes: (E_AL_A, E_BL_B, E_CL_C), (E_AL_A, E_BL_C, E_CL_B), (E_AL_B, E_BL_A, E_CL_C), (E_AL_B, E_BL_C, E_CL_A), (E_AL_C, E_BL_A, E_CL_B), (E_AL_C, E_BL_B, E_CL_A).

(a) P(only Adam right) = $\frac{1}{6}$

(b) P(none of them right) = $\frac{2}{6} = \frac{1}{3}$

(c) P(all of them right) = $\frac{1}{6}$

33. $\begin{cases} \frac{y}{24} = k, y = 24k \text{ ......(i)} \\ \frac{y+12}{24+12} = 2k, y = 72k - 12 \text{ ....(ii)} \end{cases}$

Sub. (i) into (ii), $24k = 72k - 12$, $48k = 12$, $k = \frac{1}{4}$.

Put $k = \frac{1}{4}$ into (i), $y = 24(\frac{1}{4}) = 6$.

Ans. The solutions are y = 6 and $k = \frac{1}{4}$.

34. (a) $\begin{cases} m + n = 1 \text{ ......(i)} \\ m = 4n \text{ ......(ii)} \end{cases}$ Sub. (ii) into (i), $4n + n = 1$, $5n = 1$, $\therefore n = \frac{1}{5}$

(b) Let x be the number of red marbles, $\frac{x}{y} = m = 4(\frac{1}{5})$, $x = \frac{4y}{5} = 0.8y$

Ans. The number of red marbles is 0.8y.

35. (a) $\frac{1(a)+2(7)+3(b)+4(5)}{a+7+b+5} = 2.5$, $a + 3b + 34 = 2.5(12 + a + b)$, $0.5b = 1.5a - 4$, $b = 3a - 8$

(b) $\frac{b}{12+a+b} = \frac{5}{14}$, $14b = 5(12 + a + b)$, $9b = 60 + 5a$, $9(3a - 8) = 60 + 5a$, $22a = 132$, $a = 6$
Sub. $a = 6$ into $b = 3a - 8$, $b = 3(6) - 8 = 10$. $\therefore n = 12 + a + b = 12 + 6 + 10 = 28$

36. (a) The required probability $= \frac{1}{3}$

(b) Required probability $= \frac{2}{6} = \frac{1}{3}$

(c) (i) Expected amount received $= \frac{1}{3}(7) + \frac{1}{3}(6) = \$4\frac{1}{3}$

(ii) $\$4\frac{1}{3} \neq \$5$, $\therefore$ it is not a fair game.

37. (a) (i) The points awarded = 3
(ii) The points awarded = 1 + 3 + 5 = 9
(iii) The points awarded = 1

(b) (i) The required probability $= \frac{10}{20} = \frac{1}{2}$

(ii) The numbers include: 42, 44, 46, 48, 54. The required probability $= \frac{5}{20} = \frac{1}{4}$

(c) $\frac{1}{4} \times 175 = 43.75$, $\therefore k = 44$

(d) There are 10 numbers containing "4",

$\therefore$ the expected points awarded $=\frac{1}{2}(1)+\frac{10}{20}(3)+\frac{1}{4}(5)=3.25$

38. (a) Not greater than 5:

(1,1)…(1,4), (2,1)…(2,3)…(4,1).

The required probability $=\frac{4+3+2+1}{6\times 6}=\frac{10}{36}=\frac{5}{18}$

(b) (i) The expected amount that the player is awarded $=\frac{5}{18}(30)+\frac{18-5}{18}(12)\ =\$17<\$20$

Thus, the claim is disagreed.

(ii) Expected net loss = 20 – 17 = \$3

(c) $\frac{5}{18}(30)+\frac{18-5}{18}(k)>20$, $k>16\frac{2}{13}$, $\therefore$ the least integral value of k is 17.