Unit 9 Areas & volumes (3): Pyramids, cones & spheres

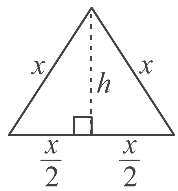
⚠ Important facts ⚠

1. Key terms

- pyramid (錐體), cone (圓錐體), sphere (球體), hemisphere (半球體), frustum (平截頭體)
- base (底), vertex (頂點), slant edge (斜邊), slant height (斜高)
- lateral surface (側面), curved surface (曲面)
- arc (弧), sector (扇形), circumference (圓周)

2. Relevant previous knowledge

(a) Equilateral triangle



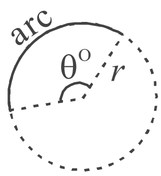
- height of the equilateral triangle

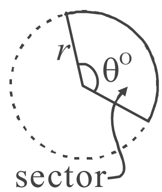
$$= h = \sqrt{x^2 - \left(\frac{x}{2}\right)^2} = \sqrt{\frac{3x^2}{4}} = \frac{\sqrt{3}x}{2}$$

- area of an equil. $\Delta = \frac{1}{2}(x)(h) = \frac{1}{2}(x)\left(\frac{\sqrt{3}x}{2}\right) = \frac{\sqrt{3}}{4}x^2$

(b) Circle

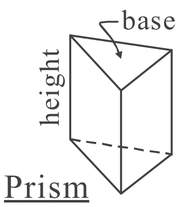
- circumference $= 2\pi r$, area $= \pi r^2$
- arc length $= 2\pi r \times \dfrac{\theta}{360}$,
- sector area $= \pi r^2 \times \dfrac{\theta}{360}$

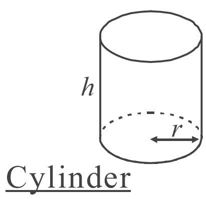




(c) Prism and cylinder

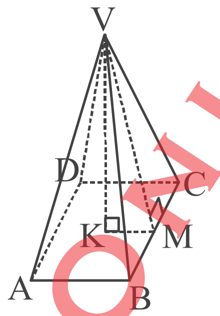
- vol. of prism = base area × h
- vol. of cylinder $= \pi r^2 \times h$
  $= \pi r^2 h$





- curved (lateral) surface area of a cylinder
  $= 2\pi\, r \times h \; = 2\pi\, r\, h$
- total surface area of a solid cylinder $= 2\pi\, r\, h + 2\pi\, r^2$

3. Pyramids



(a) In the given pyramid:
ABCD: the *base*, VK: the *height*,
V: the *vertex*, VB: a *slant edge*,
VM: a *slant height*
(i.e. the height of the lateral surface)

(b) The slant edges of a right pyramid (直立錐體) are equal.
In the above figure, VA = VB = VC = VD.

(c) Slant height (斜高) > height of the pyramid
Slant edge (斜邊) > slant height

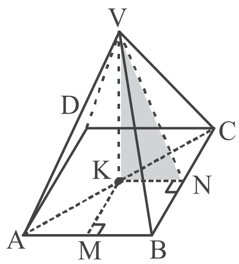
(b) Volume of a pyramid $= \frac{1}{3} \times$ base area $\times$ height

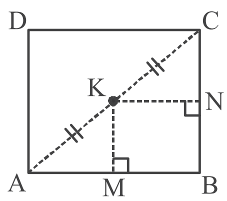
(e) If the slant edge is given, we would make use of the diagonal of the base to find the height of the pyramid.

Example: If AB = 6 cm, BC = 8cm, each slant edge = 13cm,

then: $AK = \sqrt{AM^2 + KM^2}$
$= \sqrt{3^2 + 4^2}$
$= 5$ cm

$\therefore$ the height VK
$= \sqrt{VA^2 - AK^2}$
$= \sqrt{13^2 - 5^2} = 12$ cm





In calculating the lateral surface area, we need to calculate the slant height(s) of the lateral surfaces. For example, referring to the figure in (d):

the slant height VN $= \sqrt{VB^2 - BN^2}$
$= \sqrt{13^2 - (\frac{8}{2})^2}$
$= \sqrt{153}$ cm

the slant height VM $= \sqrt{VA^2 - AM^2}$

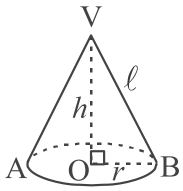
$= \sqrt{13^2 - (\frac{6}{2})^2} = \sqrt{160}$ cm

$\therefore$ the total surface area

$= 6 \times 8 + 2 \times \left(\dfrac{1}{2} \times 6 \times \sqrt{160} + \dfrac{1}{2} \times 8 \times \sqrt{153} \right)$

$= 222.8$ cm^2 (1.d.p.)

4. Cones

(a) 

In the given cone:
V: vertex, *∠AVB: vertical angle,*
r: base radius, *h: height,*
ℓ : slant height

(b) $\ell^2 = h^2 + r^2$, $h^2 = \ell^2 - r^2$, $r^2 = \ell^2 - h^2$

(c) Volume of a cone $= \dfrac{1}{3} \times$ base area $\times$ height $= \dfrac{1}{3} \pi r^2 h$

(d) The curved (lateral) surface area $= \pi r \ell$
The total surface area of a solid cone $= \pi r \ell + \pi r^2$

(e) ∠OVB is called the *semi-vertical angle*.

(f) The curved surface of a cone can be flattened as a sector whose radius is the slant height of the cone:

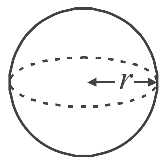
arc length = base circumference, i.e.

$2\pi \ell \times \dfrac{\theta}{360} = 2\pi r$

sector area = curved surface area, i.e.

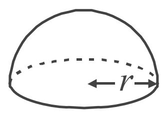
$\pi \ell^2 \times \dfrac{\theta}{360} = \pi r \ell$

5. Spheres and hemispheres



(a) Volume of a sphere $= \dfrac{4}{3} \pi r^3$

Total surface area of a sphere $= 4\pi r^2$



(b) Volume of a hemisphere $= \dfrac{1}{2} \times \dfrac{4}{3} \pi r^3 = \dfrac{2}{3} \pi r^3$

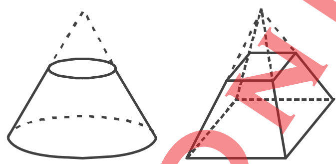
Surface area of a hollow hemisphere $= \frac{1}{2} \times 4\pi r^2 = 2\pi r^2$

Total surface area of a solid hemisphere

$= 2\pi r^2 + \pi r^2 = 3\pi r^2$

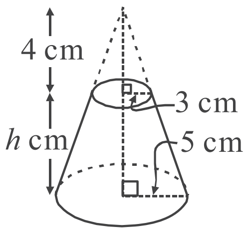
6. Frustums

(a) Frustums are the remaining part of cutting the top of a prism or a cone along a plane parallel to its base.



(b) In problems about frustums, we may need to use similar triangles to find the height of the frustum.

Example:



$\frac{h+4}{4} = \frac{5}{3}$, $\therefore$ $h+4 = \frac{5}{3} \times 4$,

$h = \frac{20}{3} - 4 = \frac{8}{3}$

(c) *More about frustums can be found in Unit 10.*

7. *See Unit 10 for similar solids.*

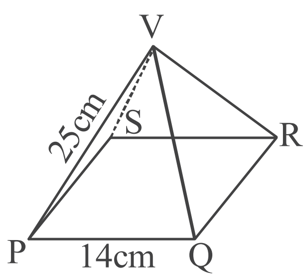
In this exercise, give the answers to 3 significant figures or 1 decimal place when appropriate.

(I) Warm-up items, No.1-26

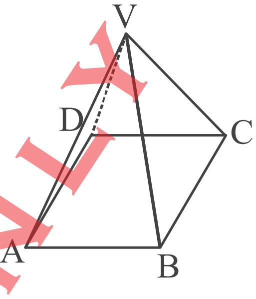
1. The figure shows a right pyramid with a square base of side 14 cm and slant edge 25 cm.

(a) Find the total surface area of the pyramid.

(b) Find the height of the pyramid. Then find the volume of the pyramid.



2. The right pyramid VABCD in the figure has a rectangular base in which AB = DC = 10 cm, BC = AD = 20 cm. The height of the pyramid is 12 cm.



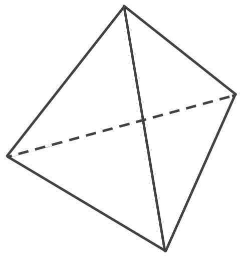
(a) Find the volume of the pyramid.

(b) Find the two slant heights of the pyramid.

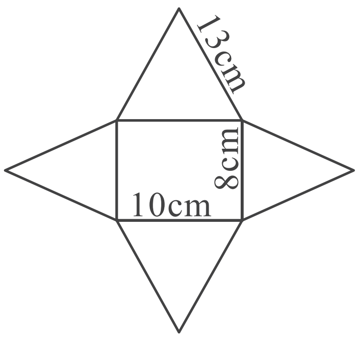
(c) Find its total surface area.

3. Find the volume of the right pyramid whose base is a rhombus of diagonals 10 cm and 24 cm, and its height is 16 cm.

4. The figure shows a regular tetrahedron of side 4cm. Find its total surface area.



5. The figure shows the net of a right pyramid which is composed of a rectangle and two pairs of isosceles triangles.



(a) Find the total surface area of the pyramid.

(b) Find the volume of the pyramid.

6. A right pyramid with a square base has a height of 15 cm and a volume of 2 000 cm^3.

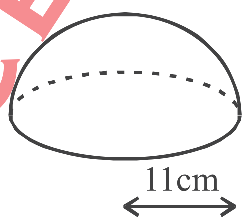
(a) Find the area of its base.

(b) Find the length of its slant edge.

(c) Find the total surface area of the pyramid.

7. Find the volume of the largest sphere that can be carved from a piece of cubical wood of side 7 cm.

8. Find the volume and total surface area of a hemisphere with radius 11cm.
(Leave π in your answer.)



9. Find the diameter of a ball with a surface area of 100 cm^2.

10. The total surface area of a sphere is 320 cm^2. Find its volume.

11. Three spheres of radii 3cm, 5cm and 7 cm are melted and recast into a large sphere.

(a) Find the radius of the new sphere.

(b) Find the percentage change in surface area.

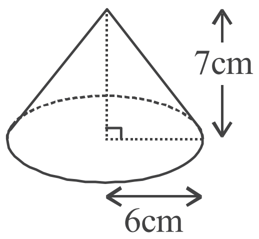
12. A solid sphere is recast into 8 identical spheres. Find the percentage change in total surface area.

13. A hollow metal sphere has an external diameter of 15 cm and thickness of 2 cm. If the metal is 150g/cm^3, find the weight of the hollow sphere.

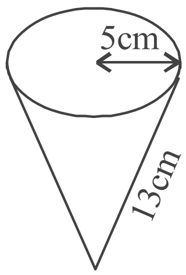
14. A cylindrical tank of radius 6 cm contains some water. 10 marbles, each of diameter $\frac{3}{4}$ cm, are dropped into the water. If the marbles are all immersed in the water and no water overflows, find the rise in water level.

15. Find the volumes and the total surface areas of the following cones.

(a)



(b)



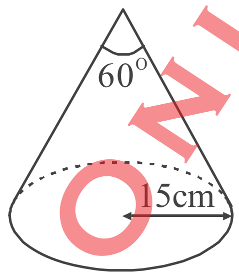
16. The height of a cone is 20 cm and the circumference of its base is 30 cm. Find its curved surface area.

17. Find the volume of the cone with curved surface area 40π cm^2 and base radius 5 cm.

18. A solid metal cylinder of radius 6 cm and height 10 cm is melted and recast into a cone with height 12 cm. Find the base radius of the new cone.

19. The volume of a solid cone is 1152π cm^3. If the length of its base radius is half that of the height, find its curved surface area in surd form. *(Leave π in your answer.)*

20. The base radius of a solid cone is 15 cm and its vertical angle is 60°. Find its total surface area and volume.

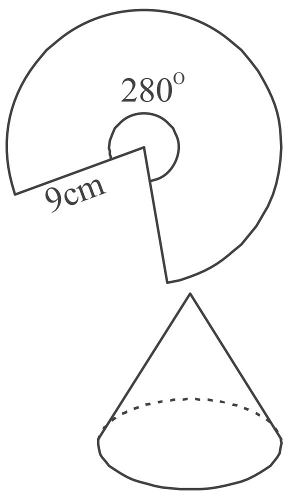


21. A semi-circular sheet of diameter 14 cm is rolled to form a cone. Find the volume of the cone.

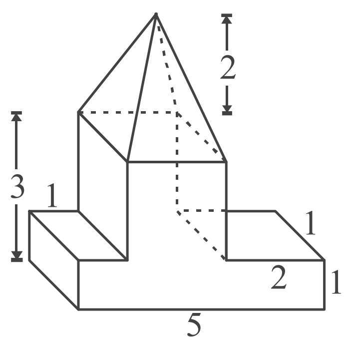
22. A piece of paper in the form of a sector is folded to form a cone. The radius of the sector is 9 cm and the angle at the centre is 280°.

(a) Find the curved surface area of the cone.

(b) Find the volume of the cone.



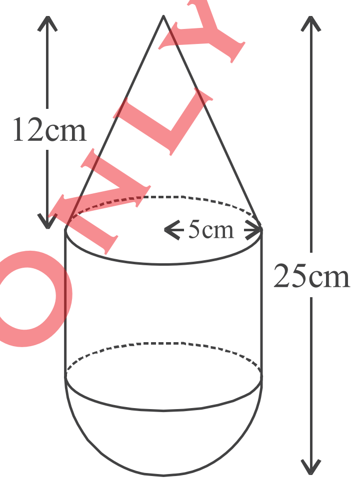
23. Find the volume of the given solid.



24. The solid in the figure is composed of a cylinder with a hemisphere attached at one end and a cone at the other end. The height of the cone is 12 cm and its base radius is 5 cm. The height of the solid is 25 cm.

(a) Find the volume of the solid.

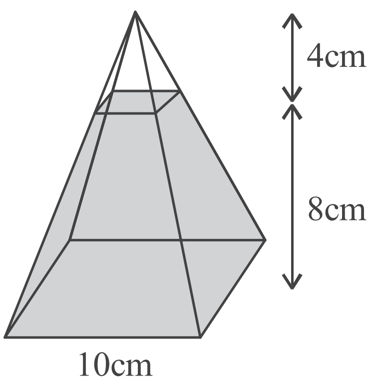
(b) Find the total surface area of the solid.



25. A frustum is formed by cutting out a similar pyramid from a pyramid whose base is a square of side 10 cm. The heights of the cut pyramid and the frustum are 4 cm and 8 cm respectively.

(a) Find the volume of the frustum.

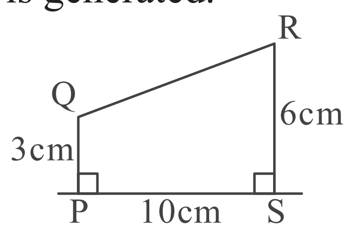
(b) Find the total surface area of the frustum.



26. PQRS is a trapezium with right angles at P and S. When it is rotated about the line PS, a frustum is generated.

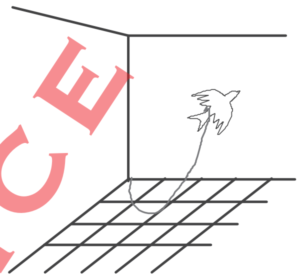
(a) Find the lateral surface area of the frustum.

(b) Find its volume.



(II) Stimulating items, No. 27-40

27. A hummingbird is tied to a string 1 metre long attached to a corner of a rectangular room where the walls and floor are at right angles to one another. The room is 2m × 3m × 3m. Find the space, in m^3, in which the humming bird can move.

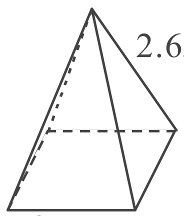


28. The figure shows a right pyramid with square base $2x$ cm and slant edge $2.6x$ cm.

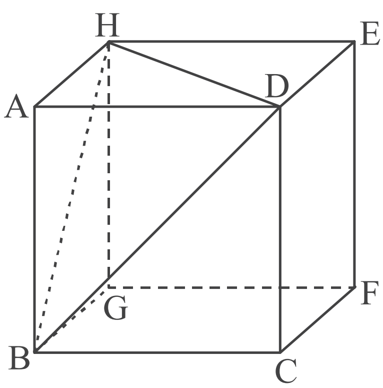
(a) Find its slant height in terms of x.

(b) If its total surface area is 1360 cm^2, find the value of x.

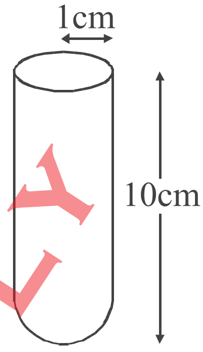
(c) Find the volume of the pyramid, correct to 1 decimal place.



29. ABCDEFGH is a cube with volume a cm^3. A tetrahedron BADH is cut away from the cube along the plane BDH. Find the volume of the remaining solid in terms of a.



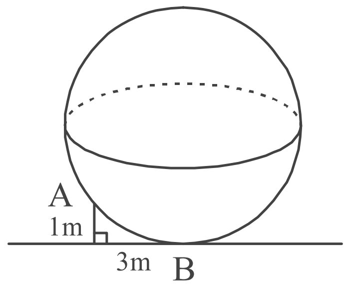
30. The figure shows a test-tube consisting of a cylindrical upper part of radius 1 cm and a hemispherical lower part of the same radius. The height of the test-tube is 10 cm



(a) Find its capacity.

(b) When two-thirds of the test-tube is filled with water, find the total area of the wet surface.

31. A rod 1 meter long touches the sphere vertically at a point A when the rod is 3 meters away from B. Find the surface area and volume of the sphere.



32. Figure A shows the outer shell of a mould which is of the form of a conical frustum. The height of the frustum is 6 cm and its base radii are 4 cm and 6 cm. Figure B shows its inner space as well, which is in the form of a cylinder together with a hemisphere. The depth of the inner space is 5 cm, and the radius of the cylindrical part is 4 cm.

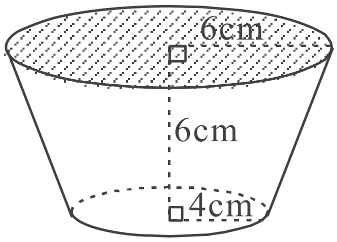


Figure A

(a) Find the capacity of the inner space of the mould in terms of π.

(b) The mould is made of wood weighs $0.8\text{g}/\text{cm}^3$. Find the weight of the mould when it is empty.

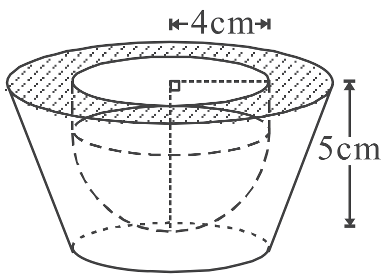
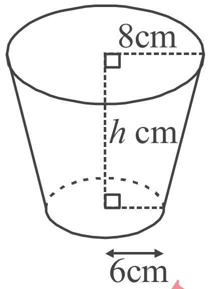


Figure B

33. The figure shows a frustum with volume 148π cm^3. The radii of the upper surface and the base are 8 cm and 6 cm respectively.

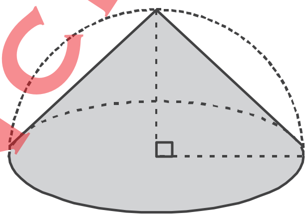


(a) Find the height of the frustum.

(b) Find the total surface area of the frustum.

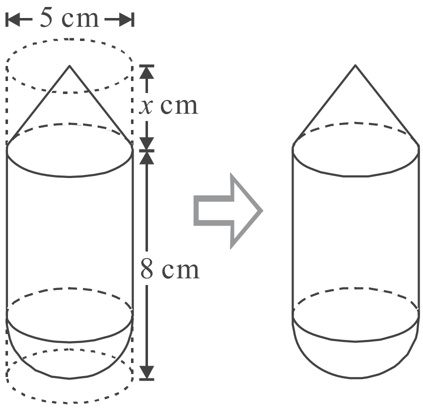
34. The figure shows a right cone carved from a hemisphere. The original hemisphere had been immersed completely in a rectangular tank of water and the water level rose 6 cm.

(a) Find the ratio of volumes of the hemisphere to the cone.



(b) What will be the rise in water level if the cone is immersed into that water tank?

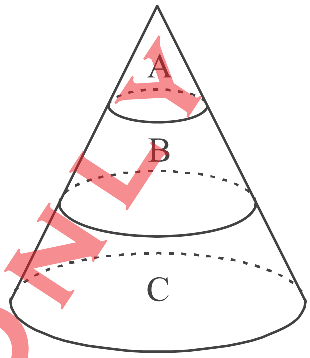
35. A sculpture is cut from a metal cylinder as shown in the figure. The sculpture consists of a circular cone at the top, a cylinder in the middle and a hemisphere at the bottom. It is found that 20% of the material has been removed from the original cylinder in order to make the sculpture.



(a) Find the value of x.

(b) If the cost of polishing a metal object is \$0.4 per cm^2, find the cost of polishing the whole surface of the sculpture.

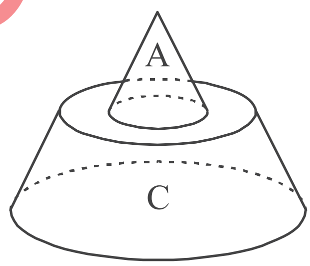
36. A cone with base radius 15 cm and height 30 cm is cut into three parts by two parallel horizontal planes. Part A is a smaller cone, while part B and part C are frustums. The upper and lower base radii of frustum B are 5 cm and 12 cm respectively.



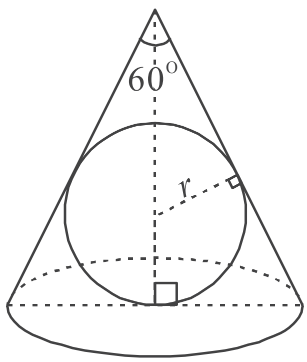
(a) Find the volume of frustum B.

(b) Find the lateral surface area of frustum B. Express the answer in surd form, and leave π in the answer.

(c) Frustum B is then removed, and the remaining parts are stuck together as shown in the figure. Find the total surface area of the solid formed.



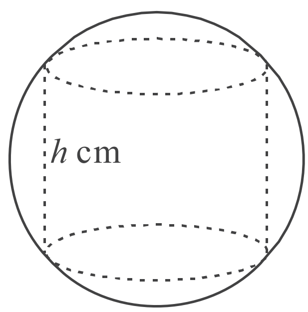
37. A sphere with radius r is inscribed in a right cone with a vertical angle of 60°.



(a) Find the height of the cone in terms of r.

(b) Find the ratio of volumes of the cone and the sphere.

38. A solid right wooden cylinder of height h cm and volume V cm^3 is carved symmetrically from a solid sphere of radius of 12cm.

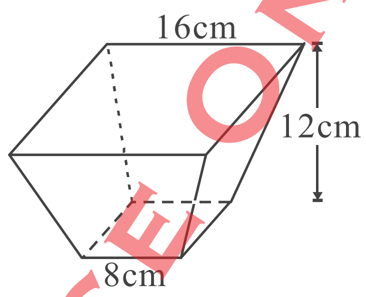


(a) Find the value of V in terms of h.

(b) It is known that the radius of the sphere is two-thirds of the height of the cylinder. Find the ratio of volume of the cylinder to that of the wood remained.

39. The figure shows an inverted hollow frustum of height 12 cm and the sides of its square bases are 8 cm and 16 cm.

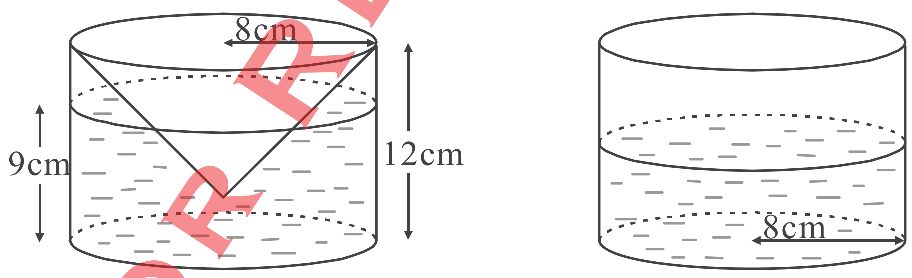
(a) Find the volume of the frustum.



(b) Water now flows through a pipe into the frustum. Find the volume of water when it reaches a height of 9 cm.

(c) If the internal radius of the pipe is 0.8 cm, and water flows at a rate of 6 cm/s, find the time it has taken to fill the frustum to the height of 9 cm.

40. An inverted cone of base radius 8 cm and height 9 cm has been put into a cylindrical vessel of the same radius and their upper bases overlaps. The water level of the vessel becomes 9 cm. If the height of the vessel is 12 cm, find the new water level after the inverted cone is removed.



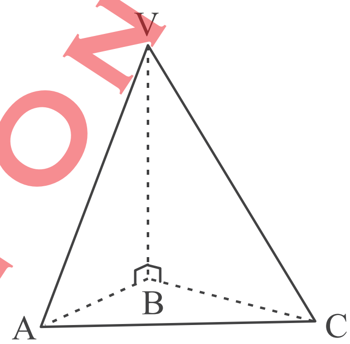
(III) Exam Practice, No. 41-43

41. Figure (a) shows a pyramid VABC. Its base is ΔABC, and its height is VB. It is given that AB = 3 cm, BC = 4 cm, AC = 5 cm and VB = 6 cm.

(a) Is ΔABC a right-angled triangle? Explain your answer.

(b) Find the volume of the pyramid VABC.

(c) Let D be a point on the face VAC such that BD is the height of the pyramid VABC. It is given that the area of ΔVAC is $3\sqrt{29}\text{ cm}^2$. Find BD in surd form.



42. The height and the capacity of a right conical paper cup are 12 cm and $100\pi\text{ cm}^3$ respectively.

(a) Find the base radius of the paper cup.

(b) The paper cup is cut along the slant edge and then a sector is obtained.

 (i) Find the radius of the sector.

 (ii) Find the angle of the sector. Give your answer correct to 1 decimal place.

43.

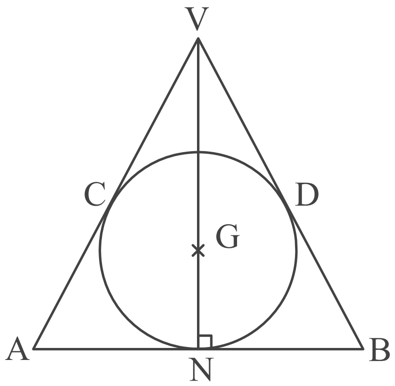


Figure (a)

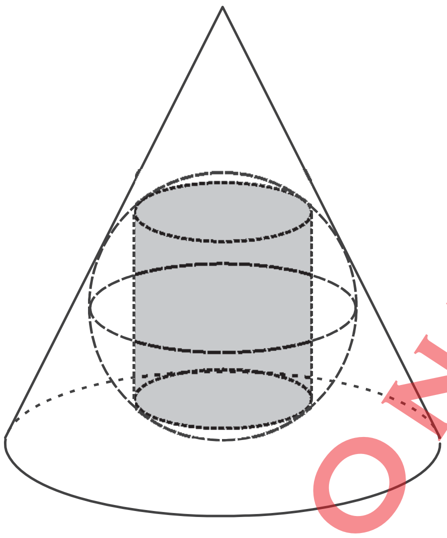


Figure (b)

(a) Figure (a) shows an isosceles triangle VAB with VA = VB. Circle CDN is inscribed in ΔVAB. The height VN of ΔVAB passes through the centre G of the circle . Prove that $\angle VCG = \angle VDG = 90°$.

(b) In Figure (b), a cylinder with base radius 4 cm and height 6 cm is inscribed in a sphere, and the sphere is inscribed in a cone with height 15 cm.

(i) Find the radius of the sphere.

(ii) Find the base radius of the cone in surd form.

(iii) Find "volume of the cone : volume of the sphere".